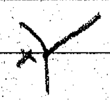
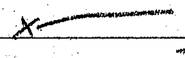
Diagrams + Symmetry Factors for the Path Integral (Partition Function) vs Correlation Functions

Let's consider the real scalar ϕ^3 theory as in Srednicki. We can work in n dimensions. Ignoring counterterms & the Z_g function, the Feynman rules for the interacting path integral are

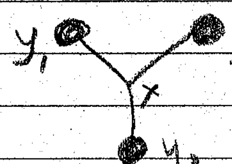
Vertex  $= \int d^n x \, (ig)$

Propagator  $= -i\,\Delta(x-y) \equiv -i\,\Delta_{xy}$

Source  $= \int d^n x \; iJ(x) \equiv \int d^n x \, iJ_x$

Symmetry factors come from interchangeable vertices, propagators, and sources.

Therefore, the 3-source, 1-vertex term in the path integral is



$$= \frac{1}{3!}\int d^n y_1 \, d^n y_2 \, d^n y_3 \, d^n x \, (ig) \, J_1 J_2 J_3 \, \Delta_{1x} \Delta_{2x} \Delta_{3x}$$

Symmetry factor is $3!$ b/c the sources are indistinguishable. Let's derive this from the path integral. This comes from

$$\int d^n x \left(\frac{ig}{3!}\right)\left(-i\frac{\delta}{\delta J(x)}\right)^3 \exp\left[\int d^n y \, d^n z \, \frac{i}{2} J_y \Delta_{yz} J_z\right]$$

with all derivatives acting on the exponential.

Each derivative pulls down a factor of

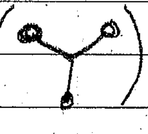
$$\int d^n y_i \; \Delta(x-y_i) J(y_i)$$

with only one way to get this term, meaning there is nothing to cancel the $1/3!$ from the exponential.

Now consider the rules for correlation functions $\langle 0|T\varphi(x_1)\varphi(x_2)\dots|0\rangle$. These are the same as for the path integral, but there are no factors for external points along with no symmetry factors — each external point is at a distinct specified location.

Continuing our example,

$$\langle T\varphi(x_1)\varphi(x_2)\varphi(x_3)\rangle = \left(-i\frac{\delta}{\delta J_3}\right)\left(-i\frac{\delta}{\delta J_2}\right)\left(-i\frac{\delta}{\delta J_1}\right)\left(\quad\right)\Big|_{J=0}$$



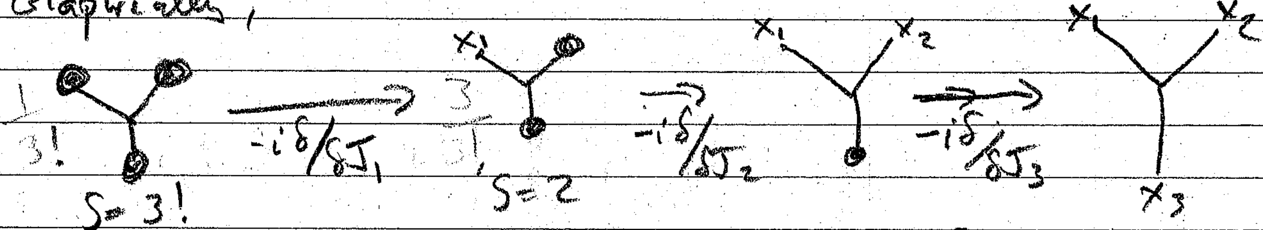
Each derivative must act on the J factors outside the exponential. The 1st can act on 1 of 3 J factors, giving a factor of 3. The 2nd has 2 choices. We get

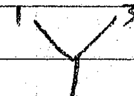
$$= (ig)\left(-i\frac{\delta}{\delta J_3}\right)\left(-i\frac{\delta}{\delta J_2}\right)\left(\frac{3}{3!}\right)(-i)\int d^n x\int d^n y_2\int d^n y_3\,\Delta_{x_1 x}\Delta_{y_2 x}\Delta_{y_3 x}J_{y_2}J_{y_3}$$

$$= -ig\left(-i\frac{\delta}{\delta J_3}\right)\left(\frac{3!}{3!}\right)\int d^n x\int d^n y_3\,\Delta_{x_1 x}\Delta_{x_2 x}\Delta_{y_3 x}J_{y_3}$$

$$= (i)\,ig\int d^n x\;\Delta_{1x}\Delta_{2x}\Delta_{3x}$$

Graphically,



Note that there is only 1 diagram b/c  also has each external point next to each other one.

Now let's look at the connected diagram w/ 2 vertices + 4 sources. (Bard's question)
From the path integral, we want the connected part of

from expanding $\exp(ig\int\varphi^3)$

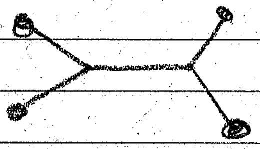
$$\frac{1}{2}\left(\frac{ig}{3!}\int d^n x\left(-i\frac{\delta}{\delta J}\right)^3\right)^2 \exp\left[\frac{i}{2}\int d^n y\int d^n z\, J(y)\Delta(y-z)J(z)\right]$$

The 1st vertex gives  as before, so we have

$$\frac{1}{2}\frac{(ig)^2}{(3!)^2}\int d^n x\int d^n y\left(-i\frac{\delta}{\delta J(x)}\right)^3\int \Delta_{z_1 y}\Delta_{z_2 y}\Delta_{z y}J_{z_1}J_{z_2}J_z\, e^{\frac{i}{2}\langle J\Delta J\rangle}\, d^n z_1\, d^n z_2\, d^n z$$

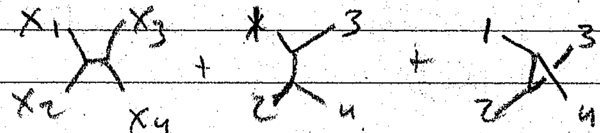
For the connected diagram w/ 4 sources one of the next set of derivatives (factor of 3) must act on 1 of the 3 J factors outside the exponential (another factor of 3). That leaves us with

$$(-i)\frac{(ig)^2}{8}\int d^n x\int d^n y\int d^n z_1\, d^n z_2\, d^n z_3\, d^n z_4\, \Delta_{1y}\Delta_{2y}\Delta_{3x}\Delta_{4x}J_1J_2J_3J_4\, e^{\frac{i}{2}\langle J\Delta J\rangle}$$

This is a single diagram  $S=8$

The symmetry factor is (1/2) from exchanging vertices ~~times~~ (1/2) for each of the left + right pairs of sources. This is the same as figure 9.10 in the book

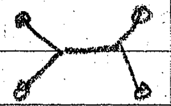
To get the correlation function, the diagrammatic rule is to consider each pairing of external points, there are 3 ways to do that, leading to 3 diagrams

$\langle T\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\rangle =$ 

There is not a symmetry factor b/c the external points are distinct + the vertices are as well (by virtue of connecting to distinct external points)

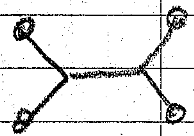
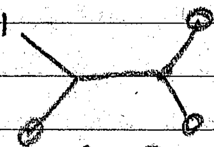
We can also differentiate the path integral diagram to get this. Let's do it graphically. Note the sums occur due to the product rule.

$\langle T\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\rangle =$

$\left(-i\frac{\delta}{\delta J_4}\right)\left(-i\frac{\delta}{\delta J_3}\right)\left(-i\frac{\delta}{\delta J_2}\right)\left(-i\frac{\delta}{\delta J_1}\right)\Big($  $Z_0\Big)\Big|_{J=0}$

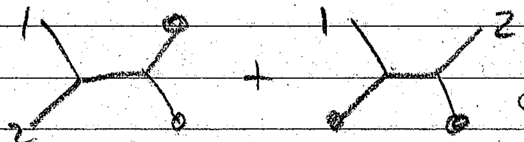
So again we must differentiate the source "blobs" on the diagram, not Z_0.

The 1st derivative comes w/a factor of 4 b/c it can differentiate either source (x2) on either vertex (x2)

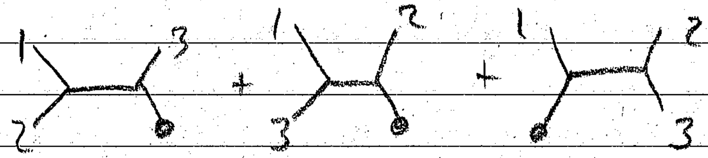
 $\xrightarrow{-i\frac{\delta}{\delta J_1}}$ 

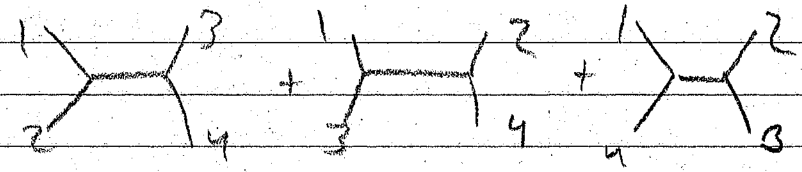
$S = 8$ $\qquad$ $S = 2$

Now we have product rule w/a factor of 2 in one spot

$\xrightarrow{-i\delta/\delta J_2}$  ← i.e. factor of 2 larger.

$S = 2$

$\xrightarrow{-i\delta/\delta J_3}$ 

$\xrightarrow{-i\delta/\delta J_4}$ 

If you put the external points in a canonical order, you get the 3 diagrams on the previous page.

To get scattering amplitudes, you just drop the propagators to the external points + Fourier transform according to LSZ.