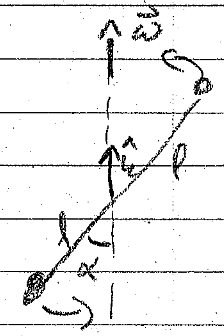
Rotation, Part II

Inertia Tensor

- A Possibly Surprising Example



  - Consider 2 particles of mass m each joined by a light rod of length $2l$
    - + The rod makes angle α with respect to $\hat{k}$ axis
    - + The rod rotates around $\hat{k}$ $\quad \vec{\omega} = \omega\hat{k}$
  - The moment of inertia around this axis is $2ml^2\sin^2\alpha$ so the angular momentum look like $\hat{k}\cdot\vec{L} = 2ml^2\sin^2\alpha\,\omega$
  - But let's calculate $\vec{L}$ directly
    - + Each particle is at position

$$\vec{r} = \pm\, l\sin\alpha\left[\cos(\omega t)\hat{\imath} + \sin(\omega t)\hat{\jmath}\right] \pm l\cos\alpha\,\hat{k}$$

so $\vec{v} = \frac{d\vec{r}}{dt} = \pm\,\omega l\sin\alpha\left[-\sin(\omega t)\hat{\imath} + \cos(\omega t)\hat{\jmath}\right]$

    - + Angular momentum is

$$\vec{L} = \sum_i m_i \vec{r}_i \times \vec{v}_i = 2m\omega l^2\left[\sin^2\alpha\,\hat{k} - \sin\alpha\cos\alpha\left(\sin(\omega t)\hat{\jmath} + \cos(\omega t)\hat{\imath}\right)\right]$$

    - + what are the extra components?

- Angular Momentum of a Rotating Object
  - Consider an object rotating around some origin with angular velocity $\vec{\omega}$
    - + We've seen each particle has velocity $\dot{\vec{r}}_i = \vec{\omega}\times\vec{r}_i$
    - + Then

$$\vec{L} = \sum_i m_i \vec{r}_i \times(\vec{\omega}\times\vec{r}_i) = \sum_i m_i\left[(\vec{r}_i^{\,2})\vec{\omega} - (\vec{r}_i\cdot\vec{\omega})\vec{r}_i\right]$$

$$\longrightarrow \int dm\left[(\vec{r}^{\,2})\vec{\omega} - (\vec{r}\cdot\vec{\omega})\vec{r}\right]$$

    - + In our previous rotation studies, we considered symmetric objects where the part of the 2nd term $\perp$ to $\vec{\omega}$ cancels. Note that the part of $\vec{L} \parallel \vec{\omega}$ is just $I\omega$ as defined before
    - + You can also check this matches the 2 particles above.

- Inertia tensor of an object
  - We can write $\vec{L}$ in components as
    $$L_a = \sum_b I_{ab}\,\omega_b \quad \text{(where } a, b \text{ are component indices)}$$
  - I_{ab} is the inertia tensor
    $$I_{ab} = \sum_i m_i\left[\vec{r}_i^{\,2}\,\delta_{ab} - r_a r_b\right] \to \int dm\left(\vec{r}^{\,2}\delta_{ab} - r_a r_b\right)$$
  - If we define coordinates w.r.t. inertial axes $\hat{i}, \hat{j}, \hat{k}$, I_{ab} changes in time as the object rotates. So we often use rotating **body axes** $\hat{x}, \hat{y}, \hat{z}$ that are fixed in the object. Then I_{ab} is a property of the object (given the origin).

- Tensors:
  - Mathematically, a (rank-2) tensor $\overleftrightarrow{T}$ is a linear transformation that turns a vector into another vector
  - If you write a vector as a column, a tensor is a square matrix
    - Elements of the matrix are components of the tensor, meaning tensors have 2 indices
    - The tensor acts by matrix multiplication
      $$\overleftrightarrow{T}\cdot\vec{a} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ \vdots & \ddots & \vdots \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$
    - In components, that's
      $$\left(\overleftrightarrow{T}\cdot\vec{a}\right)_a = \sum_b T_{ab}\,a_b$$
  - If you rotate axes, components change $[\vec{a}] \to R[\vec{a}]$ and $[\overleftrightarrow{T}] \to R[\overleftrightarrow{T}]R^{-1}$ which means $[\overleftrightarrow{T}\cdot\vec{a}] \to R[\overleftrightarrow{T}\cdot\vec{a}]$, like a vector should.

- Then $\vec{L} = \overleftrightarrow{I}\cdot\vec{\omega}$

- Dynamics of Rotation
  - A reminder of some things we already learned
    - When $\vec{L}$ measured in an inertial frame
    - For a [illegible], $d\vec{L}/dt = \vec{\tau}$ external torque
    - For uniform gravity, $\vec{F} = M\vec{g}$ and $\vec{\tau} = M\vec{R}\times\vec{g}$ where M = total mass, $\vec{R}$ = CM position

- For purely rotational motion, $\vec{v}_i = \vec{\omega} \times \vec{r}_i$ for each particle.
  + Then kinetic energy is

$$T = \frac{1}{2} \sum_i m_i (\vec{\omega} \times \vec{r}_i)^2 = \frac{1}{2} \sum_i m_i \, \underbrace{\vec{\omega} \cdot (\vec{r}_i \times (\vec{\omega} \times \vec{r}_i))}_{\text{triple product}}$$

$$= \frac{1}{2} \vec{\omega} \cdot \vec{L} = \frac{1}{2} \vec{\omega} \cdot (\overleftrightarrow{I} \cdot \vec{\omega})$$

  + Since dot products don't change when you rotate axes,

- If $\vec{r}_i{}'$ is the position of a component particle relative to CM,

$$\sum_i m_i \vec{r}_i{}' = 0 \quad (\text{a } \int dm \, \vec{r}\,' = 0)$$

From this, we can prove (by writing $\vec{r} = \vec{R} + \vec{r}\,'$)

  + <u>Parallel Axis Theorem</u> The inertia tensor around some origin is

$$I_{ab} = (I_{cm})_{ab} + M(R^2 \delta_{ab} - R_a R_b)$$

  where $\vec{R}$ = CM position and $\overleftrightarrow{I}_{cm}$ = inertia tensor for rotations around the CM

  + For general motion

$$\vec{L} = M \vec{R} \times \frac{d\vec{R}}{dt} + \overleftrightarrow{I}_{cm} \cdot \vec{\omega}$$

$$T = \frac{1}{2} M \left(\frac{d\vec{R}}{dt}\right)^2 + \frac{1}{2} \vec{\omega} \cdot (\overleftrightarrow{I}_{cm} \cdot \vec{\omega})$$

  Means we can separate out CM motion from rotation around the CM. Recall uniform gravity acts on CM

— Principal Axes

- Components of the inertia tensor (work with body axes)
  + The diagonal components are <u>moments of inertia</u>

  For ex,

$$I_{zz} = \int dm \, (r^2 - z^2) = \int dm \, (x^2 + y^2)$$

  + For any axis given by unit vector $\hat{n}$, the moment of inertia is $\hat{n} \cdot \overleftrightarrow{I} \cdot \hat{n}$
  + The off-diagonal components are <u>products of inertia</u>

$$I_{xy} = -\int dm \, xy$$

- Example: A uniform density cube of side length ℓ sits with corner at the origin. What is $\overleftrightarrow{I}$?
  + Due to axis permutation symmetry, we only need to find I_{zz}, I_{xy}

+ The moments of inertia are
$$I_{xx} = I_{yy} = I_{zz} = \int_0^l dx \int_0^l dy \int_0^l dz \left(\frac{M}{l^3}\right)(x^2+y^2) = \frac{2}{3}Ml^2$$

+ The products of inertia are
$$I_{xz} = I_{yz} = I_{xy} = -\int_0^l dx \int_0^l dy \int_0^l dz \left(\frac{M}{l^3}\right) xy = -\frac{1}{4}Ml^2$$

+ As a matrix,
$$I = Ml^2 \begin{bmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{bmatrix}$$

+ By the parallel axis theorem, $\vec{R} = (l/2)\hat{x} + (l/2)\hat{y} + (l/2)\hat{z}$

$$(I_{cm})_{zz} = I_{zz} - M(x_{cm}^2 + y_{cm}^2) = \frac{1}{6}Ml^2, \quad (I_{cm})_{xy} = I_{xy} + Mx_{cm}y_{cm} = 0$$

$\overleftrightarrow{I}_{cm}$ is a diagonal matrix

- In general, $\vec{L} = \overleftrightarrow{I}\cdot\vec{\omega}$ is not parallel to $\vec{\omega}$.
  + That's just because a matrix × a vector is not ∝ that vector usually.
  + But suppose $\vec{\omega}$ is an eigenvector of $\overleftrightarrow{I}$. Then
    $\overleftrightarrow{I}\cdot\vec{\omega} = I\vec{\omega}$ where I is the eigenvalue.

  + Mathematically, a vector $\vec{a}$ is an eigenvector of square n×n matrix M if $\vec{a} \neq 0$ and $M\cdot\vec{a} = \lambda\vec{a}$ for a number λ called the eigenvalue.
  + To find the eigenvalues, note that $(M - \lambda \mathbb{1})$ (where $\mathbb{1}$ = identity) is not invertible. That gives the characteristic eqn

$$\det(M - \lambda\mathbb{1}) = 0 \leftarrow \text{polynomial equation for } \lambda$$

Each solution is eigenvalue $\lambda_1, \dots, \lambda_n$ where M is n-dim.

  + You can then find the eigenvector for eigenvalue λ_1 ($\vec{a}_1$) by solving
$$(M - \lambda_1\mathbb{1})\vec{a} = 0$$ and a normalization condition.
If 2 eigenvalues are the same, there is a 2D subspace of eigenvectors, so you can choose 2 orthogonal ones, etc.

- The principal axes of an object are the orthonormal (eigenvectors) $\hat{e}_1, \hat{e}_2, \hat{e}_3$ of the inertia tensor

+ These are usually taken to be body axes, but they can be inertial axes in a symmetric situation

+ The eigenvalues of $\overleftrightarrow{I}$ corresponding to $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are I_1, I_2, I_3 and called <u>principal</u> <u>moments</u>: In this basis, $\overleftrightarrow{I}$ is a diagonal matrix.

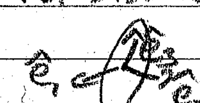
+ Because $\overleftrightarrow{I}$ is a real symmetric matrix, we can choose the 3 principal axes to be orthogonal. You can prove the principal moments are positive

+ For $\vec{\omega} = \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3$, $\vec{L} = I_1\omega_1 \hat{e}_1 + I_2 \omega_2 \hat{e}_2 + I_3 \omega_3 \hat{e}_3$ and $T = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + I_3 \omega_3^2$

- Symmetric objects and principal axes

  → Normally body axes b/c of rotation of object, but an object rotating around an axis of symmetry keeps shape

  + If an object has a symmetry, that can tell you principal axes. + For ex, an axis of rotational symmetry is a principal axis, + any 2 ⊥ unit vectors in the ⊥ plane are also principal axes w/ same principal moments. This is a <u>symmetric object</u>



  + Any time you calculate + find 2 (or 3) equal principal moments, any axes in that subspace are principal! See the cube

$$\overleftrightarrow{I}_{cm} = \frac{1}{6} M l^2 \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$$

so any vector is an eigenvector! A rectangular prism w/ 1 side a square has any ~~axis~~ in the square principal.

- **Describing motion from a body frame**

  This is convenient b/c the inertia tensor is constant

  ~~- Euler's equations~~

  - Euler's Equations

    - As measured in an inertial frame $\frac{d\vec{L}}{dt} = \vec{\tau}$

    - A body frame rotates with the object's angular velocity $\vec{\omega}$

$$\Rightarrow \frac{d\vec{L}}{dt} = \dot{\vec{L}} + \vec{\omega} \times \vec{L} = \vec{\tau}$$

inertial frame ↗ ↑ rotating frame

+ Let's use principal axes, as above.
In components, $\dot{L}_1 = I_1 \dot{\omega}_1$, etc
and $(\vec{\omega} \times \vec{L})_1 = \omega_2 I_3 \omega_3 - \omega_3 I_2 \omega_2$

+ Therefore

$$\left.\begin{aligned} I_1 \dot{\omega}_1 + \omega_2 \omega_3 (I_3 - I_2) &= \tau_1 \\ I_2 \dot{\omega}_2 + \omega_3 \omega_1 (I_1 - I_3) &= \tau_2 \\ I_3 \dot{\omega}_3 + \omega_1 \omega_2 (I_2 - I_1) &= \tau_3 \end{aligned}\right\} \text{Euler's equations}$$

• These are easy to use if $\vec{\tau} = 0$ or if torque is constant direction in body frame (from drag, thrusters, etc)

– Motion of a Free Symmetric Top

• A symmetric top has $I_1 = I_2 \equiv I$, and it is free if $\vec{\tau} = 0$. Let's look at free motion
• If $I = I_3$ also, Euler's eqns are $\dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0$.
Rotation is constant!

• If $I \neq I_3$,
+ $I_3 \dot{\omega}_3 = 0 \Rightarrow \omega_3$ is constant
+ And

$$I \dot{\omega}_1 = (I - I_3) \omega_3 \omega_2, \quad I \dot{\omega}_2 = (I_3 - I) \omega_3 \omega_1$$

+ Because $\omega_3 = $ const, we can write

$$\dot{\omega}_1 = \Omega \omega_2, \quad \dot{\omega}_2 = -\Omega \omega_1, \quad \Omega = (I - I_3)\omega_3 / I$$

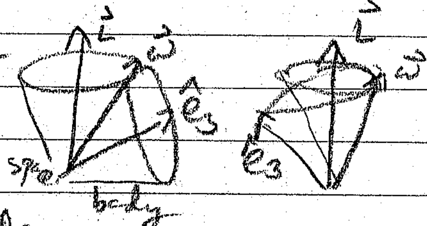
+ If we take another derivative

$$\ddot{\omega}_1 = -\Omega^2 \omega_1 \Rightarrow \omega_1 = A \cos(\Omega t + \phi)$$
$$\Rightarrow \omega_2 = -A \sin(\Omega t + \phi)$$

• Wobble of the top: what does this mean?
+ $\vec{\omega}$ rotates around $\hat{e}_3$ (precession called a wobble) with frequency $|I - I_3| \omega_3 / I$. Direction depends on if $I < I_3$ or $I > I_3$. Angle between $\vec{\omega}$ and $\hat{e}_3$ is const.
+ In an inertial frame $\vec{L} = $ constant is conserved. But $I_3 \omega_3 = \hat{e}_3 \cdot \vec{L} = $ const also, so $\hat{e}_3$ rotates around $\vec{L}$ at constant angle.

+ Kinetic energy $T = \frac{1}{2}\vec{\omega}\cdot\vec{L}$ = const is conserved, so $\vec{\omega}$ is at a fixed angle from $\vec{L}$

+ If we draw this, $\vec{\omega}$ traces out a cone called the space cone around $\vec{L}$ and the body cone around $\hat{e}_3$



+ We need to learn more to derive details

• Earth's Chandler wobble b/c $I - I_3 \simeq -I/300$, Really takes longer than 300 days. small amplitude

— Stability of rotation

• Suppose we have a general object mostly rotating around one principal axis $\hat{e}_3$ (a whichever)

+ In general, I_1, I_2, I_3 are all different.

+ $\vec{\omega} = \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3$ and $\omega_3 \gg \omega_1, \omega_2$ by assumption

+ Still assume torque-free motion.

• Euler equations are ~~$I_3 \dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2 \approx 0$~~

the $I_3 \dot{\omega}_3 = (I_1 - I_2)\omega_1 \omega_2 \approx 0$, so ω_3 = const approximately.

+ Then

$$I_1 \dot{\omega}_1 = (I_2 - I_3)\omega_3 \omega_2, \quad I_2 \dot{\omega}_2 = (I_3 - I_1)\omega_3 \omega_1$$

are coupled again.

+ By differentiating + substituting

$$I_2 I_1 \ddot{\omega}_1 = (I_2 - I_3)(I_3 - I_1)\omega_3^2 \omega_1 \quad \text{etc}$$

• What does this mean? Take our guess $\omega_1 = Ae^{\beta t}$

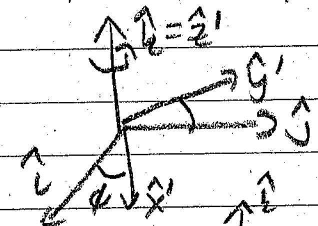
+ If ($I_3 > I_1$ and $I_3 > I_2$) or ($I_3 < I_1$ and $I_3 < I_2$) we see $\beta^2 < 0$, so the solution for ω_1 (and ω_2) is oscillatory + stays small. This is stable.

+ But if $I_1 < I_3 < I_2$ or $I_2 < I_3 < I_1$, then $\beta^2 > 0$. After a little while, $\omega_1 \propto e^{+\beta t}$ grows exponentially. This is unstable.

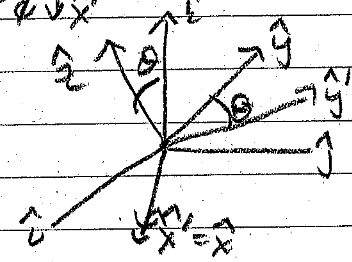
+ An object will rotate around the axis w/ largest or smallest principal moment but will not stay rotating around the middle one

⊙ Euler Angles: Description of orientation + rotation

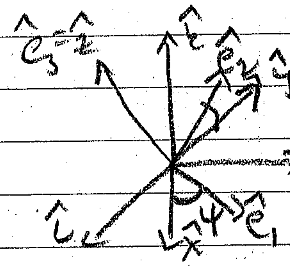
- Orientation of a rigid object
  - Requires 3 angles to describe orientation
    - + We will use Euler's angles
    - + You can use quaternions (generalized complex numbers)
  - Euler angles ϕ, θ, ψ
    - + Line up principal axes $\hat{e}_1, \hat{e}_2, \hat{e}_3$ with inertial axes $\hat{i}, \hat{j}, \hat{k}$.
    - + Rotate by ϕ around $\hat{k}$.
      Principal axes now line up with $\hat{x}', \hat{y}', \hat{z}' = \hat{k}$



    - + Rotate by θ around $\hat{x}'$.
      Principal axes now line up with $\hat{x}, \hat{y}, \hat{z}$.
      The $\hat{x} = \hat{x}'$ axis is called the line of nodes. (Note: some books rotate by θ around $\hat{y}' = \hat{y}$, so there is a slight difference)



    - + Rotate by ψ around $\hat{z}$. This gives final $\hat{e}_1, \hat{e}_2, \hat{e}_3$ position with $\hat{e}_3 = \hat{z}$.



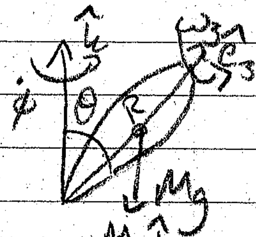
  - The rotation
    - + The object rotates with angular velocity
      $\vec{\omega} = \dot{\phi}\hat{k} + \dot{\theta}\hat{x} + \dot{\psi}\hat{e}_3$
      (Note: we're now just using dots as d/dt blah, these angles are just functions, not vectors)
    - + We need this in terms of principal axes.
      So note
      $\hat{z} = \hat{e}_3$, $\hat{x} = \cos\psi\, \hat{e}_1 - \sin\psi\, \hat{e}_2$, $\hat{y} = \sin\psi\, \hat{e}_1 + \cos\psi\, \hat{e}_2$
      and $\hat{k} = \cos\theta\, \hat{z} + \sin\theta\, \hat{y}$.
    - + By taking dot products to find components

$$\vec{\omega} = (\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\hat{e}_1 + (\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)\hat{e}_2 + \underbrace{(\dot{\phi}\cos\theta + \dot{\psi})}_{= \omega_3}\hat{e}_3$$

– Precession

- If you like, it's possible to work out more details of free precession. But let's look at precession due to gravity again

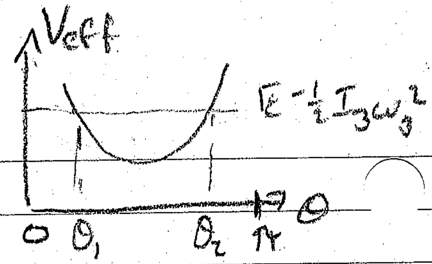
- We have a symmetric top $I_1 = I_2 = I$ spinning around $\hat{e}_3$. Due to gravity, it precesses around $\hat{z}$.



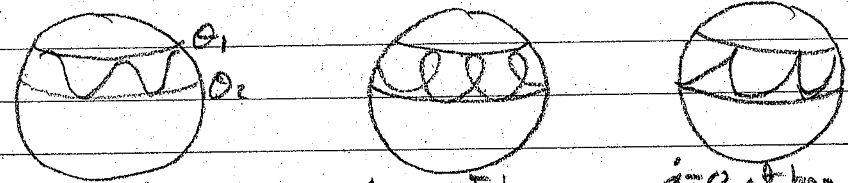
  + CM is position $R\hat{e}_3$. Gravity has force $-Mg\hat{z}$. Torque is therefore $\vec{\tau} = -MgR\,(\hat{e}_3 \times \hat{z})$.
  + If it did not precess, ω_3 would be $\dot{\psi}$. But $\dot{\phi}$ mixes in b/c $\hat{e}_3 \cdot \hat{z} = \cos\theta \neq 0$ (Euler angles)
  + Precession rate is $\dot{\phi}$ b/c that describes rotation of $\hat{e}_3$

- There are 3 conserved quantities
  + Total energy $E = \frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{I}\cdot\vec{\omega} + MgR\cos\theta$
  + Since $\vec{\tau} \perp \hat{z}$, $L_z = \hat{z}\cdot\vec{L} = I_3\omega_3\cos\theta + I\dot{\phi}\sin^2\theta = \text{const}$
  + Also $\tau_3 = \hat{e}_3\cdot\vec{\tau} = 0$, so Euler eqn. is
  $$I_3\dot{\omega}_3 + (I - I)\omega_1\omega_2 = I_3\dot{\omega}_3 = \tau_3 = 0$$
  $\Rightarrow \omega_3 = \text{const}$.
  + Note: L_z and $I_3\omega_3$ overlap b/c $\hat{z}\cdot\hat{e}_3 \neq 0$. But together, they give $\dot{\phi} = (L_z - I_3\omega_3\cos\theta)/I\sin^2\theta$ in terms of θ

- Effective Potential
  + We have
  $$T = \tfrac{1}{2}I(\omega_1^2 + \omega_2^2) + \tfrac{1}{2}I_3\omega_3^2 = \tfrac{1}{2}I\dot{\theta}^2 + \tfrac{1}{2}I\sin^2\theta\,\dot{\phi}^2 + \tfrac{1}{2}I_3\omega_3^2$$
  + We can write the total
  $$E = T + V = \tfrac{1}{2}I_3\omega_3^2 + \tfrac{1}{2}I\dot{\theta}^2 + \left[\frac{1}{2}\frac{(L_z - I_3\omega_3\cos\theta)^2}{I\sin^2\theta} + MgR\cos\theta\right]$$
  + Define $V_{eff}(\theta)$ = quantity in square brackets. Then
  $$E - \tfrac{1}{2}I_3\omega_3^2 = \text{const} = \tfrac{1}{2}I\dot{\theta}^2 + V_{eff}(\theta)$$



- Nutation
  - Generally, θ ranges between values θ_1 and θ_2 where
    $E - \frac{1}{2} I \omega_3^2 = V_{eff}(\theta)$
  - This bobbing motion while spinning + precessing is <u>nutation</u>
  - Has 3 patterns depending on $L_z - I_3 \omega_3 \cos(\theta_1)$, ie, $\dot\phi$ at the top



$\dot\phi > 0$ always $\dot\phi < 0$ at top $\dot\phi = 0$ at top ← case if you hold + let it go

- Steady Precession is when θ = constant
  - Requires $E - \frac{1}{2} I_3 \omega_3^2 = \min(V_{eff})$
  - $\frac{dV_{eff}}{d\theta} = I_3 \omega_3 \sin\theta\, \dot\phi - I \sin\theta\, \dot\phi^2 \cos\theta - MgR\sin\theta = 0$
    b/c $L_z - I_3 \omega_3 \cos\theta = I \sin^2\theta\, \dot\phi$
    $$\Rightarrow \dot\phi = \frac{I_3 \omega_3}{2 I \cos\theta} \pm \frac{(I_3^2 \omega_3^2 - 4 MgRI\cos\theta)^{1/2}}{2 I \cos\theta}$$
  - Steady precession is only possible for
    $$I_3^2 \omega_3^2 \geq 4 MgRI\cos\theta$$
    You have to be spinning fast enough!
  - If $I_3^2 \omega_3^2 \gg 4MgRI$, you can expand the square root
    $$\dot\phi \approx I_3\omega_3 / I\cos\theta \quad \text{or} \quad \dot\phi \approx MgR / I_3\omega_3 \quad \text{small}$$
  - We previously found the 2nd <u>slow precession</u> solution. The 1st is <u>fast precession</u> + is rare.