

# Rotation, Part I

## ⊙ Rigid Objects

### - Properties of rigid bodies

- A rigid body is a collection of a number of particles held together w/ positions  $\vec{r}_i$ 
  - + The distance between two particles in a rigid body is fixed:  $|\vec{r}_i - \vec{r}_j| = \text{constant}$
  - + Rigid bodies cannot flex or bend. They can have an overall linear motion (translation), and they can rotate.

### • Center of mass position

- + The center of mass (CM) position  $\vec{R}$  is the mass-weighted average position

$$\vec{R} = \frac{1}{M} \sum_i m_i \vec{r}_i \quad \text{where } M = \sum_i m_i = \text{total mass}$$

- + Total momentum of the object is

$$\vec{P} = \sum_i \vec{p}_i = \sum_i m_i \dot{\vec{r}}_i = \frac{d}{dt} \left( \sum_i m_i \vec{r}_i \right) = M \dot{\vec{R}}$$

- + This is the same as a point object of mass  $M$  at the CM position

- + For a continuous object, replace sums by integrals over density

$$M = \int d^3\vec{r} \rho(\vec{r}) \equiv \int dm, \quad \vec{R} = \frac{1}{M} \int d^3\vec{r} \rho(\vec{r}) \vec{r} \equiv \frac{1}{M} \int dm \vec{r}$$

### • Rotating around a fixed axis

- + Say the object rotates around the  $z$ -axis. Each component particle has  $z = \text{const}$  and circular motion in  $x_i, y_i$

+ In cylindrical

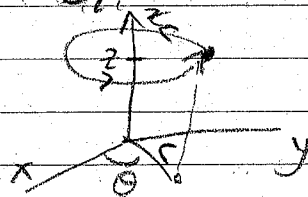
That  $\vec{r}_i = (r_i \cos \theta_i) \hat{i} + (r_i \sin \theta_i) \hat{j} + z_i \hat{k}$

+ For the object to be rigid,

$\dot{r}_i = 0$  for each particle

and  $\dot{\theta}_i = \omega$  is the same for

all the parts of the object (it is rigid)



+ As usual, we define the angular velocity vector pointing along the axis of rotation.

In our case,  $\vec{\omega} = \omega \hat{k}$ . We can check

$$\dot{\vec{r}}_i = \vec{\omega} \times \vec{r}_i \text{ for every particle}$$

+ Check that distances between particles are constant.

$$\frac{1}{2} \frac{d}{dt} (|\vec{r}_i - \vec{r}_j|^2) = (\dot{\vec{r}}_i - \dot{\vec{r}}_j) \cdot (\vec{r}_i - \vec{r}_j) = [\omega \times (\vec{r}_i - \vec{r}_j)] \cdot (\vec{r}_i - \vec{r}_j) = 0$$

• Moment of Inertia

+ Using cylindrical coordinates, each particle has velocity  $\vec{v}_i = r_i \omega \hat{\theta}$ , so the z-component of angular momentum is  $L_{z,i} = m_i r_i^2 \omega$

+ The total  $L_z = \sum_i m_i r_i^2 \omega \rightarrow \left( \int dm r^2 \right) \omega$

+ The quantity in parentheses is moment of inertia around the axis

$$I \equiv \int dm r^2$$

where  $r$  = distance from the axis

- Integrating in curvilinear coordinates

Can use Jacobian determinant, but let's take a more intuitive approach

• Plane Polar Coordinates

+ How do we measure distance?

Infinitesimal displacement along  $\hat{r}$  is  $dr$ ,  
 along  $\hat{\theta}$  is  $r d\theta$ . Pythagoras says  
 $ds^2 = dr^2 + r^2 d\theta^2$

+ This matches what we have for speed

$$v^2 = (ds/dt)^2 = \dot{r}^2 + r^2 \dot{\theta}^2$$

+ But these distances are the sides of a rectangular cell. So

$$\int d^2\vec{r} f(\vec{r}) = \int dr d\theta r f(r, \theta)$$

+ Cylindrical coordinates in 3D are an easy generalization

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

$$\int d^3\vec{r} f(\vec{r}) = \int dz \int dr \int d\theta r f(r, \theta, z)$$

### • Spherical Polar Coordinates

+ These are coordinates sit.

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

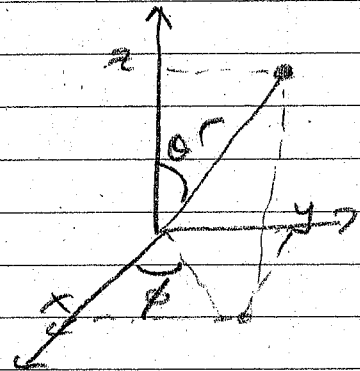
$$z = r \cos\theta$$

+ That means

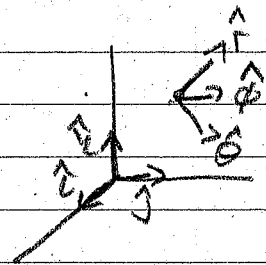
$r$  = distance from origin

$\theta$  = angle between  $\vec{r}$  and  $z$  axis ( $\hat{z}$ )

$\phi$  = angle from  $x$  axis ( $\hat{x}$ ) to projection of  $\vec{r}$  in the  $(xy)$ -plane



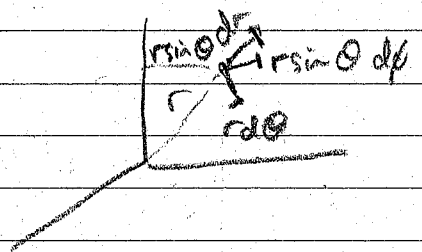
+ There are unit vectors pointing in the directions of increasing coordinates. Note that  $\hat{\phi}$  is  $\perp \hat{z}$



+ The infinitesimal distances for changes  $dr$ ,  $d\theta$ ,  $d\phi$  are

$dr$ ,  $r d\theta$ ,  $r \sin\theta d\phi$ , so

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$



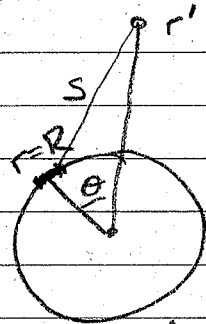
+ These distances mean 3D volume integration is

$$\int d^3r f(\vec{r}) = \int dr \int d\theta \int d\phi r^2 \sin\theta f(r, \theta, \phi)$$

• Example of spherical coord integration: gravitational potential energy near a spherical shell

+ The shell has mass  $M$  spread uniformly over a sphere of radius  $r=R$ .

+ We want the potential energy of a mass  $m$  at point  $r'$  along the  $z$  axis



+ The potential energy due to the little mass  $dM$  at position  $(\theta, \phi)$  on the shell is

$$dV = -Gm dM / s$$

where  $s^2 = R^2 + (r')^2 - 2r'R \cos\theta$  (law of cosines)

+ The total potential energy is the integral

$$V = -Gm \int \frac{dM}{s} = -Gm \int \frac{(MR^2 d\theta d\phi \sin\theta / 4\pi R^2)}{s}$$

$$= -\frac{GMm}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi \frac{d\theta \sin\theta}{[R^2 + (r')^2 - 2r'R \cos\theta]^{1/2}}$$

$$= -\left(\frac{GMm}{2}\right) \int_{-1}^1 \frac{du}{[R^2 + (r')^2 - 2r'R u]^{1/2}} \quad u = \cos\theta$$

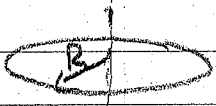
$$= -\frac{GMm}{2} \left(\frac{-1}{r'R}\right) \left[ R^2 + (r')^2 - 2r'R u \right]^{1/2} \Big|_{u=-1}^1$$


$$= \frac{-GMm}{2r'R} (|R+r'| - |R-r'|)$$

$$= -GMm \begin{cases} 1/r' & \text{for } r' > R \\ 1/R & \text{for } r' < R \end{cases}$$

+ This shows that the potential energy due to a spherical distribution of mass is the same as from a point mass (if you are outside)

## - Example Moment of Inertia Calculations

- Circular hoop of mass  $M$  and radius  $R$   
with axis  $\perp$  hoop through the center. 
- + Each mass element of the hoop is at the same distance from the axis  $\Rightarrow$  integral becomes multiplication. Note  $z=0$  everywhere,  $r=R=\text{const.}$
- + Result is  $I = MR^2$


- Thin Uniform Rod of mass  $M$ , length  $L$   
+ Axis at end point  $\perp$  rod given 

$$I_{\text{end}} = \int_0^L dx \left(\frac{M}{L}\right) x^2 = \frac{1}{3} ML^2 \quad (z, \theta = \text{const.})$$

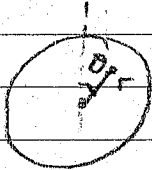
- + Axis through center of rod  $\perp$  rod

$$I_{\text{cm}} = \int_{-L/2}^{L/2} dx \left(\frac{M}{L}\right) x^2 = \frac{1}{12} ML^2$$

- Thin uniform disk of mass  $M$ , radius  $R$

- + Let's take the axis through the center  $\perp$  the disk. Using cylindrical coords w/  $z=0$ , 

$$I = \int_0^R dr \int_0^{2\pi} d\theta r \left(\frac{M}{\pi R^2}\right) r^2 = \frac{1}{2} MR^2$$

- + Axis through center but in disk: 

Use plane polar coords. in disk, so distance from axis  $= r \sin \theta$ .

$$\text{Then } I = \int_0^R dr \int_0^{2\pi} d\theta r \left(\frac{M}{\pi R^2}\right) (r \sin \theta)^2 = \frac{1}{4} MR^2$$

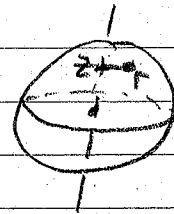
b/c we can use

$$\int_0^{2\pi} d\theta \sin^2 \theta = \frac{1}{2} \int_0^{2\pi} d\theta (1 - \cos(2\theta)) = \pi$$

• Uniform density sphere, mass  $M$ , radius  $R$   
with axis through the center.

+ In cylindrical coordinates,

$$I = \left( \frac{M}{4\pi R^3/3} \right) \int_{-R}^R dz \int_0^{\sqrt{R^2-z^2}} dr \int_0^{2\pi} d\theta r (r^2)$$

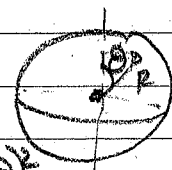


$$= \frac{3M}{8R^3} \int_{-R}^R dz (R^2 - z^2)^2 = \frac{3M}{8R^3} \left( 2R^5 - \frac{4}{3}R^5 + \frac{2}{5}R^5 \right)$$

$$= 2MR^2/5$$

This is a little annoying  
+ In spherical coordinates,

$$I = \left( \frac{M}{4\pi R^3/3} \right) \int_0^R dr \int_0^\pi d\theta \int_0^{2\pi} d\phi r^2 \sin\theta (r \sin\theta)^2$$



$$= \frac{3M}{2R^3} \left( \frac{4}{5}R^5 \right) \int_0^\pi d\theta \sin^3\theta \quad \text{Use } u = \cos\theta$$

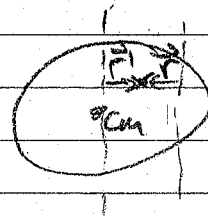
$$\text{so } \int_0^\pi d\theta \sin^3\theta = \int_{-1}^1 du (1-u^2) = 2 - \frac{2}{3} = \frac{4}{3}$$

$$\Rightarrow I = 2MR^2/5$$

• Parallel Axis Theorem

+ Suppose you have an axis through the CM of an object and another a distance  $L$  away that is parallel

+ In cylindrical coords, look at the vector to a point in the  $(x,y)$  plane from each axis.



If these are  $\vec{r}$  and  $\vec{r}'$  (both latter from CM axis) with  $\vec{L}$  the vector from the CM axis to the other, we get

$$\vec{r} = \vec{r}' - \vec{L}$$

+ Therefore, the moment of inertia for the other axis is

$$I = \int dm r^2 = \int dm (\vec{r}'^2 - 2\vec{L} \cdot \vec{r}' + L^2)$$

+ However  $\vec{L} = \text{constant}$  and  $\int dm \vec{r}' = 0$  b/c CM position for CM origin.

$$\Rightarrow I = \int dm (r')^2 + \int dm L^2 = I_{cm} + ML^2$$

+ This is the parallel axis theorem.

+ Check it works for the 2 thin rod calculations

## ● Motion Around a Single Axis

### — Dynamics of Rotation

• Change of Angular momentum

+ The total angular momentum is the sum over all component particles

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i \vec{v}_i$$

+ So then

$$\dot{\vec{L}} = \sum_i \vec{r}_i \times \vec{F}_i \quad \text{where } \vec{F}_i = \text{force on particle } i$$

But the force is

$$\vec{F}_i = \vec{F}_{i, \text{ext}} + \sum_j \vec{F}_{ij}$$

= external force plus forces from the other particles in the object

+ By Newton's 3<sup>rd</sup> law  $\vec{F}_{ij} = -\vec{F}_{ji}$  so.

$$\sum_{i,j} \vec{r}_i \times \vec{F}_{ij} = \frac{1}{2} \sum_{i,j} (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij}$$

+ If  $\vec{F}_{ij}$  is central, it is  $\propto (\vec{r}_i - \vec{r}_j)$ , so the cross product = 0. Then

$$\dot{\vec{L}} = \sum_i \vec{r}_i \times \vec{F}_{i, \text{ext}} = \vec{\tau}_{\text{ext}} = \text{external torque}$$

+ Change in angular momentum is due to external torque, i.e. ang. mom. is conserved if  $\vec{\tau}_{\text{ext}} = 0$ .  
No spontaneous rotation!

• Uniform Gravity acts on the CM

+ For  $\vec{F}_G = m\vec{g}$ , gravity on a rigid object is

$$\vec{F}_G = \sum_i m_i \vec{g} = M \vec{g}$$

+ The torque due to gravity is

$$\vec{\tau}_{\text{ext}} = \left( \sum_i m_i \vec{r}_i \right) \times \vec{g} = M \vec{R} \times \vec{g}$$

+ Force and torque act at the CM position on a mass M

+ Motion is CM in gravity + torque-free rotation around CM

• Kinetic Energy

+ Suppose the object rotates around an axis through origin with no other motion. Then (in cylindrical coords)

$$T = \frac{1}{2} \sum_i m_i \vec{v}_i^2 = \frac{1}{2} \sum_i m_i (\vec{r}_i \omega \hat{\theta})^2 = \frac{1}{2} \left( \sum_i m_i r_i^2 \right) \omega^2 = \frac{1}{2} I \omega^2$$

+ What if the object is translating + rotating around a particular axis through the CM?

Let  $\vec{r}_i = \vec{R} + \vec{r}'_i$  so  $\vec{r}'_i$  = position relative to CM

Then

$$T = \frac{1}{2} \sum_i m_i (\dot{\vec{R}} + \dot{\vec{r}}'_i)^2 = \frac{1}{2} M \dot{\vec{R}}^2 + \dot{\vec{R}} \cdot \left( \sum_i m_i \dot{\vec{r}}'_i \right)$$

$$+ \frac{1}{2} \sum_i m_i \dot{\vec{r}}'^2 = \frac{1}{2} M \dot{\vec{R}}^2 + 0 + \frac{1}{2} I_{\text{cm}} \omega^2$$

a 2nd term vanishes b/c avg. position relative to CM = 0 and 3rd term is same as above.

- Physical Pendulum

• A physical pendulum is an object hung from a pivot that can rotate around an axis through the pivot

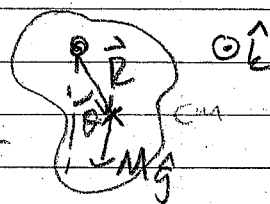
+ If  $\vec{R}$  = CM position, torque is

$$\vec{\tau} = M \vec{R} \times \vec{g} = -MRg \sin \theta \hat{k}$$

+ Angular momentum around axis is  $\vec{L} = I \dot{\theta} \hat{k}$

so

$$\dot{\vec{L}} = \vec{\tau}_{\text{ext}} \Rightarrow I \ddot{\theta} = -MRg \sin \theta \approx -MRg \theta$$



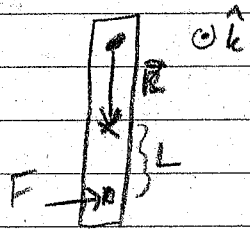


+ This is harmonic motion like a regular pendulum, but replace the frequency  $\sqrt{g/l} \rightarrow \sqrt{MRg/I}$

• "Baseball Bat Theorem"

+ An object pivots around the origin w/ CM a distance  $R$  away.

A force  $\vec{F}$  strikes the object at



a distance  $L$  farther from the CM. (Take  $\vec{F} \perp \vec{v}_{CM}$ )

+ Assuming straight line + right angles  $\vec{p} = \vec{F} + \vec{F}_{pivot}$

Moment  $\vec{L} = (F(R+L))\hat{k} = I\omega\hat{k}$  (Note  $\vec{L}$  is about pivot)

+ Total force can include a force from the pivot point

+ w/ purely rotational motion  $MR\omega$  (CM motion)

$$\vec{F} + \vec{F}_{pivot} = MR\omega\hat{k}$$

+ When is  $F_{pivot} = 0$ ? Divide eqns w/  $F_{pivot} = 0$ .

$$R+L = I/MR$$

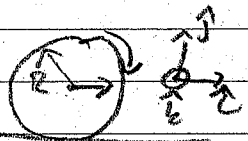
Using parallel axis theorem,  $I = I_{cm} + MR^2$

$$\Rightarrow I_{cm} = MRL$$

+ For fixed pivot position,  $L$  gives location of "sweet spot"

- Rolling

• Rolling without slipping is when a circular object (hoop, disk, cylinder) moves



forward and rotates so that the contact point is still

+ If CM velocity is  $\vec{v}$  and angular velocity  $-\omega\hat{k}$ ,

the contact point velocity is  $(v - \omega R)\hat{i} \Rightarrow v = \omega R$

for no-slip

+ The velocity of the top point is  $2\vec{v}$ , etc.

Generally, velocity is not tangent to circle

+ Any friction is static & does no work b/c

there is no motion of contact point.

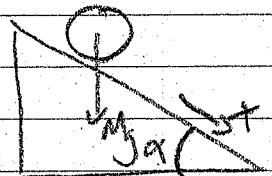
+ You can consider this as rotation around contact point

• Example: Rolling on incline

+ Linear motion has

$$M\ddot{x} = Mg \sin\theta - f_s$$

← note static friction!



+ From CM point of view, friction also leads to torque

$$I_{cm} \ddot{\theta} = F_s R$$

and with  $R\dot{\theta} = \dot{x}$ , we get  $(M + I_{cm}/R^2) \ddot{x} = Mg \sin \alpha$

+ Acceleration is slower due to rolling. Why?

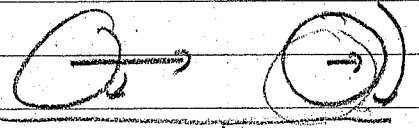
Change in potential energy is the same, but KE is

$$T = \frac{1}{2} M v^2 + \frac{1}{2} I_{cm} \omega^2 = \frac{1}{2} (M + I_{cm}/R^2) v^2$$

which is larger at same speed

• Rolling with slipping

+ Either the translation is too fast for the rotation or rotation is too fast for the translation



+ There is kinetic friction at the contact.

+ Friction speeds up or slows down rolling until slipping stops

- Precession: when the axis can move a little

• We have an object - often called a top - rotating quickly around an axis that can move slowly around a point

+ We have  $\vec{L} = I\vec{\omega} = I\omega(\cos \alpha \hat{r} + \sin \alpha \hat{k})$

so  $\dot{\vec{L}} = I\omega \cos(\alpha) \dot{\theta} \hat{\theta}$  assuming  $\omega \approx \text{const.}$

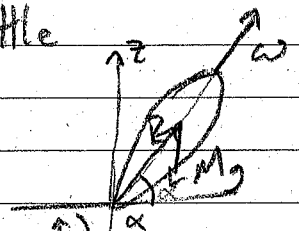
+ The torque due to gravity is

$$\vec{\tau} = \vec{R} \times M\vec{g} = MgR \cos \alpha \hat{\theta} = \dot{\vec{L}}$$

so  $\vec{L}$  rotates slowly w/ angular velocity

$$\dot{\theta} = MgR / I\omega \leftarrow \text{precession}$$

+ This analysis really only works when  $\dot{\theta} \ll \omega$  since it is an additional rotation (ie,  $\vec{\omega}$  is not quite what we said here).



• The torque from gravity causes the top to precess, not fall over

+ This is why bicycles don't fall over while you ride.

+ A small torque from sun + moon cause earth's axis to precess once every ~26,000 yr.  
"precession of equinoxes"