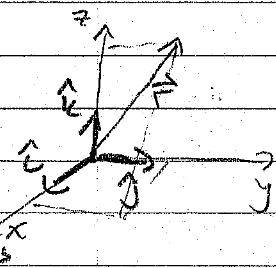
Newtonian Mechanics

- Newtonian Mechanics describes most of everyday experience, especially with EM added on.
  - It works when things are not too fast (relativity) or too small (quantum) or with strong gravity (GR)
  - Mathematically it describes the world with vectors that obey differential equations

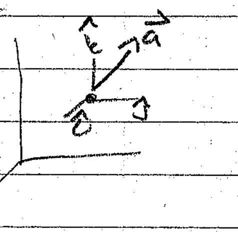
- Review of Vectors



  - Quantities with magnitude + direction
    - We usually visualize a vector extending from an origin w/ components along a set of orthogonal unit vectors or axes

      + Can write in components or a sum over the unit vectors
        $$\vec{r} = (x, y, z) = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$
      + If the components are displacements (units of length), these vectors describe positions

      + More generally vectors can start at any point

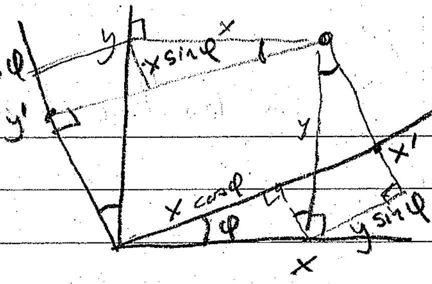


        + Key to "head to tail" addition but normally add by components
          $$\vec{a} + \vec{b} = (a_1 + b_1)\hat{\imath} + (a_2 + b_2)\hat{\jmath} + (a_3 + b_3)\hat{k}$$
    - Changing axes
      + If you choose a different set of unit vectors (say rotated axes), the vector is the same but components are different
        $$\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k} = a_1'\hat{\imath}' + a_2'\hat{\jmath}' + a_3'\hat{k}'$$

      + We can work out how components of position change when we rotate axes — the same is true for <u>all</u> vectors, not just position vectors.

Rotation around z axis by φ

$$x' = x\cos\varphi + y\sin\varphi$$
$$y' = -x\sin\varphi + y\cos\varphi$$
$$a_1' = a_1\cos\varphi + a_2\sin\varphi$$
$$a_2' = -a_1\sin\varphi + a_2\cos\varphi$$



Can you we

+ Can always relate components by a matrix multiplication

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- Vector multiplication

• Scalar/Dot Product

+ We define $\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1$, $\hat{\imath}\cdot\hat{\jmath} = \hat{\imath}\cdot\hat{k} = \hat{\jmath}\cdot\hat{k} = 0$ with commutative property, so
$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$
This is automatically also distributive.

+ For a vector from the origin, Pythagorean theorem
$$|\vec{r}|^2 = x^2 + y^2 + z^2 = \vec{r}\cdot\vec{r} = \vec{r}^{\,2}$$
$\Rightarrow$ square of vector = square of its magnitude

+ Then law of cosines says
$$|\vec{a} - \vec{b}|^2 = \vec{a}^{\,2} + \vec{b}^{\,2} - 2\vec{a}\cdot\vec{b} = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\alpha \Rightarrow \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\alpha$$
where α = angle between the vectors.

• Cross Product (Vector Product)

+ Define $\vec{a}\times\vec{b}$ = vector directed by right-hand rule: Sweep fingers of right hand from direction of $\vec{a}$ to direction of $\vec{b}$. Then thumb is in direction of $\vec{a}\times\vec{b}$.

+ This works if we define it as
$$\hat{\imath}\times\hat{\imath} = \hat{\jmath}\times\hat{\jmath} = \hat{k}\times\hat{k} = 0, \quad \hat{\imath}\times\hat{\jmath} = -\hat{\jmath}\times\hat{\imath} = \hat{k},$$
$$\hat{\jmath}\times\hat{k} = -\hat{k}\times\hat{\jmath} = \hat{\imath}, \quad \hat{k}\times\hat{\imath} = -\hat{\imath}\times\hat{k} = \hat{\jmath}.$$

+ Therefore

$$\vec{a}\times\vec{b} = (a_2b_3-a_3b_2)\hat{\imath} + (a_3b_1-a_1b_3)\hat{\jmath} + (a_1b_2-a_2b_1)\hat{k}$$

$$= \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{bmatrix}$$

+ You can see that $\times$ is distributive and anti commutative

$$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}, \quad \vec{a}\times\vec{a} \equiv 0$$

+ If you choose $\vec{a}$ along $\hat{\imath}$ and $\vec{b}$ at angle α in (x,y) plane, you see $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\alpha$

+ Triple product cyclicly symmetric $(\vec{a}\times\vec{b})\cdot\vec{c} = (\vec{b}\times\vec{c})\cdot\vec{a}$, etc and vector triple product satisfies "BAC - CAB rule"

$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$$ <u>not</u> associative

- Vector Calculus

• Suppose you have a vector function of a variable t

$$\vec{a} = a_1(t)\hat{\imath} + a_2(t)\hat{\jmath} + a_3(t)\hat{k}$$

+ Assume the unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ are constants. We will see the non-constant case soon

+ Derivatives + integrals act on each component separately

$$\frac{d\vec{a}}{dt} = \frac{da_1}{dt}\hat{\imath} + \frac{da_2}{dt}\hat{\jmath} + \frac{da_3}{dt}\hat{k}$$

$$\vec{a}(t) = \int dt\,\frac{d\vec{a}}{dt} = \left(\int dt\,\frac{da_1}{dt}\hat{\imath}\right) + \ldots$$

+ Vector products obey product rule if you keep the order

$$d/dt\,(\vec{a}\cdot\vec{b}) = d\vec{a}/dt\cdot\vec{b} + \vec{a}\cdot d\vec{b}/dt$$

$$d/dt\,(\vec{a}\times\vec{b}) = d\vec{a}/dt\times\vec{b} + \vec{a}\times d\vec{b}/dt$$

• <u>Gradient</u>: Consider a 3D function $f(\vec{r})$

+ We can assemble partial derivatives into a vector function called the gradient

$$\vec{\nabla} f = (\partial f/\partial x)\hat{\imath} + (\partial f/\partial y)\hat{\jmath} + (\partial f/\partial z)\hat{k}$$

- $\vec{\nabla} f$ points in the direction of greatest increase
  $f(\vec{r}+\Delta\vec{r}) = f(\vec{r}) + \left(\partial f/\partial x \, \Delta x + \dots\right) + \dots = f(\vec{r}) + \Delta\vec{r}\cdot\vec{\nabla} f(\vec{r})$

- If position is a function of time, the chain rule is
$$\frac{d}{dt} f(\vec{r}(t)) = \frac{d\vec{r}}{dt}\cdot\vec{\nabla} f$$

• Divergence & Curl
- Let $\vec{A}(\vec{r})$ be a vector function of position, i.e., it has 3 components
- Divergence = "gradient dot $\vec{A}$" = $\vec{\nabla}\cdot\vec{A}$
  where $\vec{\nabla} = \hat{\imath}\,\partial/\partial x + \hat{\jmath}\,\partial/\partial y + \hat{k}\,\partial/\partial z$
- Curl = "gradient cross $\vec{A}$" = $\vec{\nabla}\times\vec{A}$
- Because partial derivatives commute for smooth functions
$$\vec{\nabla}\times\vec{\nabla} f = 0, \quad \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$$

• Integrals
- To integrate a function (or vector) over a volume, integrate each coordinate sequentially
$$\int d^3\vec{r}\, f = \int dx\left(\int dy\left(\int dz\, f(\vec{r})\right)\right)$$
  or other order, depending on limits
- You can convert from Cartesian to other coordinates, which we will see soon.
- For a path $\vec{r}(t)$ and a vector function $\vec{A}$, the line integral along the path
$$\int d\vec{r}\cdot\vec{A} \equiv \int dt\left(\frac{d\vec{r}}{dt}\cdot\vec{A}\right)$$

+ There is also a <u>surface integral</u> $\int d\vec{S}\cdot\vec{A}$
where $d\vec{S} = dA\,\hat{n}$ with dA = area element, $\hat{n}$ = unit normal.

+ Two theorems:

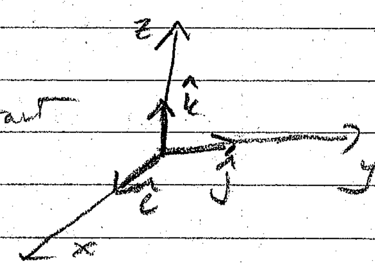
Fundamental theorem $\int d\vec{r}\cdot\vec{\nabla} f = \Delta f$

Stoke's theorem $\underset{\text{closed path}}{\oint} d\vec{r}\cdot\vec{A} = \underset{\text{any surface bounded by path}}{\int} d\vec{S}\cdot(\vec{\nabla}\times\vec{A})$

● <u>Kinematics</u> = Description of motion

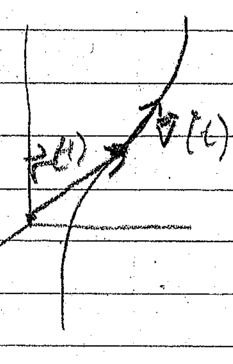
- We want to describe the motion of objects
  - Start with the motion of <u>particles</u>, infinitesimal (size, not mass), point-like object
    + Particles are featureless & described only by position
    + Many times, we can treat larger objects like particles

- Trajectory in time
  - We have to specify a <u>reference frame</u>, which is a choice of origin + axes



    + We will start with fixed/constant Cartesian axes (usually x, y, z) w/ unit vectors $\hat{i}, \hat{j}, \hat{k}$ as before
    + We follow right-hand-rule
    + Usually - not always - we choose $\hat{k}$ (+z axis) up
    + There is some time coordinate t.
  - Position, Velocity, acceleration
    + Position $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ is a function of time described as a vector from the origin to the particle's location

May sometimes use $\vec{x}$ for position



    + Define $\hat{r} \equiv \vec{r}/|\vec{r}|$ = unit vector
    + <u>Velocity</u> is change in time

Use dot for d/dt

$\vec{v}(t) = d\vec{r}/dt \equiv \dot{\vec{r}}$. By Taylor series, this is vector starting from the position

$$\vec{r}(t+\Delta t) = \vec{r}(t) + \vec{v}(t)\Delta t + \ldots$$

+ Acceleration is rate of change of velocity

$$\vec{a} = \dot{\vec{v}} = \ddot{\vec{r}}$$

• Motion with constant acceleration

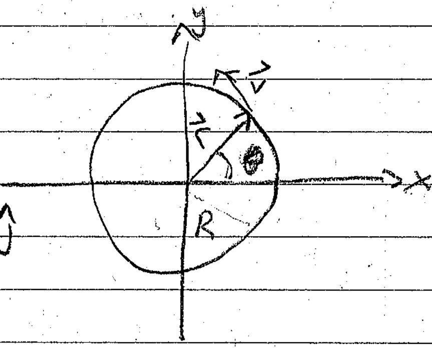
Integrate in time, vectorially

+ $\vec{v}(t) = \int dt\, \vec{a} = \underbrace{\vec{v}_0}_{\text{initial}} + \vec{a}t$

Can you show $\vec{v}_{avg} = (\vec{v}_0 + \vec{v}_{final})/2$?

+ $\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$
+ As in all cases, remember integration is not just multiplication!

• Circular Motion (in a plane)



+ Position given by angle $\theta(t)$ around circle as function of time

$$\vec{r} = R\cos(\theta(t))\hat{i} + R\sin(\theta(t))\hat{j}$$

+ Velocity is

$$\vec{v} = -R\dot{\theta}\sin(\theta(t))\hat{i} + R\dot{\theta}\cos(\theta(t))\hat{j}$$

Speed is $|\vec{v}| = R|\dot{\theta}|$, $\vec{v} \perp \vec{r}$ always

+ $\dot{\theta} \equiv$ angular velocity. Also written $\omega \equiv \dot{\theta} = |\vec{v}|/R$.

Can define a vector angular velocity out of the plane of motion s.t. $\vec{v} = \vec{\omega} \times \vec{r}$.

Constant speed is $\omega =$ const, $\theta = \omega t$.

+ Acceleration is

$$\vec{a} = \left[-R\ddot{\theta}\sin(\theta(t))\hat{i} + R\ddot{\theta}\cos(\theta(t))\hat{j}\right] \leftarrow \text{tangential, changes speed}$$
$$+ \left[-R\dot{\theta}^2\cos(\theta(t))\hat{i} - R\dot{\theta}^2\sin(\theta(t))\hat{j}\right]$$

2nd line is $= -(|\vec{v}|^2/R)\hat{r}$ always

This is centripetal acceleration directed inward along radius to keep motion circular

+ Another way to see this: $\vec{r}\cdot\vec{r} = R^2 = \text{const} \Rightarrow$
$\vec{r}\cdot\vec{v} = 0$, so $\vec{v} \perp \vec{r}$ ($\vec{v}$ tangent). Next derivative
$\Rightarrow \vec{v}^2 + \vec{r}\cdot\vec{a} = 0 \Rightarrow \hat{r}\cdot\vec{a} = -\vec{v}^2/|\vec{r}|$

+ Position $\vec{r} = R\hat{r}$ and $\vec{v} \perp \vec{r}$. Let's define $\vec{v} \equiv v\hat{\theta}$ where $\hat{\theta}$ is another unit vector. Note that this means $d\hat{r}/dt \propto \hat{\theta}$ and centripetal acceleration comes from $d\hat{\theta}/dt$.
$\vec{r} = R\hat{r}$, $\vec{v} = \omega R\hat{\theta}$, $\vec{a} = -(v^2/R)\hat{r} + \dot{v}\hat{\theta} = -\omega^2 R\hat{r} + \dot{\omega}R\hat{\theta}$

Dynamics: what determines motion?

— Newton's Laws

- First Law: "An object stays in the same state of motion unless acted on".
  + This actually depends on the reference frame: inertial (1st Law is true) vs accelerating (axes are moving, so objects can just accelerate)
  + Better way of putting 1st law: "Inertial frames exist" or "we can describe the change of motion by action of something on the object"
    ← another object
  + We will talk about accelerating frames later
- 2nd Law: More quantitative version $\vec{F} = m\vec{a}$
  + This is why we don't usually care about $d^3\vec{r}/dt^3$, etc
  + m = (inertial) mass, a property of the particle
  + Obviously we'll have to study forces!
  + We can define momentum $\vec{p} = m\vec{v}$ for a particle. Then $\vec{F} = \dot{\vec{p}}$ if m = constant. For objects made of multiple particles, this is more useful.
- 3rd Law: "Every action has an equal but opposite reaction"
  + The force of particle #1 on particle #2 is negative

the force of #2 on #1, ie $\vec{F}_{1,2} = -\vec{F}_{2,1}$
+ Total momentum is conserved (constant)
$\frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = \vec{F}_{2,1} + \vec{F}_{1,2} = 0$
if there are no external forces.

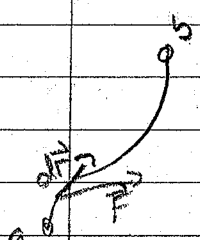
• Relativity Principle: Almost like Law 1.5
"Laws of physics are the same in all inertial frames"
+ No matter where you choose origin of space or time, no matter how you orient axes, no matter if frames move, the forces on a particle are the same in 2 frames if they are both inertial

+ This is about laws of physics, not circumstances. (See cosmology.) But basically no where/when is special in terms of physical law

— Energy

• Define Kinetic Energy $T = \frac{1}{2} m \vec{v}^2$ for a particle
(some people use K, but T is traditional)

+ Change in KE is $\frac{dT}{dt} = m\vec{v}\cdot\vec{a} = \vec{F}\cdot\vec{v} \equiv P$
The power of the force



+ Total change $\Delta T = \int P\, dt = \int \vec{F}\cdot\frac{d\vec{r}}{dt}dt = \int d\vec{r}\cdot\vec{F}$

$\equiv W$ the work done by the force on the object.
Integral is evaluated on particle's trajectory

• Potential energy: Suppose $\vec{F} = -\vec{\nabla} V(\vec{r})$
where $V(\vec{r})$ = potential energy

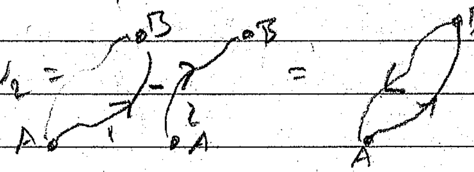
+ This means force depends only on position
+ Then the total energy $E = T + V$
is conserved

$$\frac{dE}{dt} = \frac{d}{dt}(T+V) = \vec{F}\cdot\vec{v} + \frac{d\vec{r}}{dt}\cdot\vec{\nabla}V = \vec{F}\cdot\vec{v} - \vec{F}\cdot\vec{v} = 0$$

$\Rightarrow$ This is a conservative force.

+

+ Because $\vec{\nabla}\times\vec{\nabla}V = 0$, $\vec{\nabla}\times\vec{F} = 0 \Rightarrow$ work done by $\vec{F}$ from point A to B is independent of path

$W_1 - W_2 =$  $= \oint d\vec{r}\cdot\vec{F} = \int d\vec{S}\cdot(\vec{\nabla}\times\vec{F}) = 0$

+ There are other forces like the Lorentz force $\vec{F} = q\,\vec{v}\times\vec{B}$ which conserve kinetic energy b/c they do no work $\vec{F}\cdot\vec{v} = 0$. May talk about these in Adv. Mech.