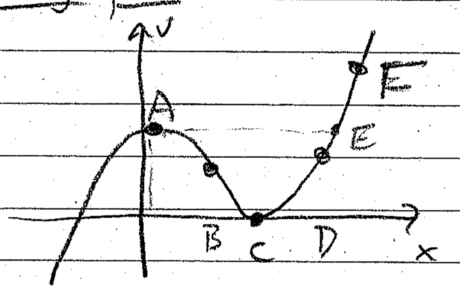
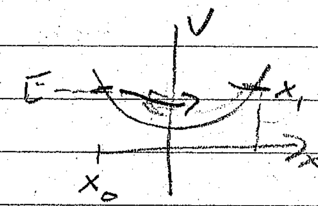
Sample Forces + Systems

- Equations of Motion
  - Once forces are specified, Newton's 2^{nd} law is a 2^{nd} order ODE
    - Called the equation of motion (EOM)
      - + Requires 2 initial conditions, typically initial position + velocity
      - + In 3D, can be coupled, so we'll try to simplify
    - If $\vec{F} = \vec{F}(\vec{r})$, solve as ODE or possibly use conservation of energy
    - If $\vec{F} = \vec{F}(t)$, $\Delta\vec{p} = m(\vec{v}_f - \vec{v}_i) = \int_{t_i}^{t_f} dt\, \vec{F}(t)$
      - + You can get $\vec{v}(t)$, integrate to get $\vec{r}(t)$
      - + $\Delta\vec{p}$ is called the impulse
    - If $\vec{F} = \vec{F}(\vec{v})$, you can convert to 1^{st} order
      $\dot{\vec{v}} = \vec{F}(\vec{v})/m$
      - + If it's a 1D problem, solve by separation of variables $\int \frac{dv}{F(v)} = \int \frac{dt}{m}$ $(t - t_0)$
      - + Then integrate $v(t)$
  - Conservative Forces Using Potential Energy
    - Find conserved E from initial conditions
      - + KE $T = \frac{1}{2}m\dot{\vec{r}}^2 \geq 0 \Rightarrow$
      - + Particle moves between turning points where $E = V(\vec{r})$
      - + What sort of behavior do you expect if the particle starts w/ $v=0$ at the given points for this 1D potential?



- More quantitatively, we get speed as a function of position
  - + It can be in any direction
  - + In 1D, $v(x) = \pm\sqrt{2(E-V(x))/m}$



  - + Then along 1 direction of travel

$$t - t_0 = \int_{x_0}^{x} dx' \sqrt{\frac{m/2}{E-V(x')}}$$

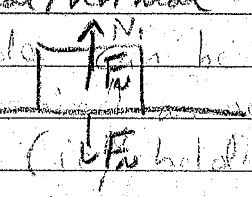
  - + The full period of motion is twice the time between turning points

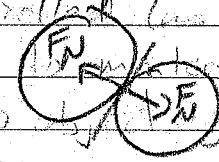
$$T = 2\int_{x_0}^{x_1} dx' \sqrt{\frac{m/2}{E-V(x')}}$$

Important Forces

Contact Forces: Forces between objects or particles that are touching

→ Normal Force: a push between two objects orthogonal/normal to surface of contact

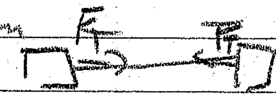




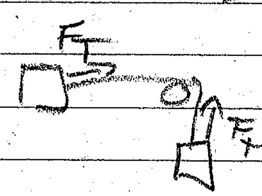
- Magnitude can be controlled/given in problem or respond to other forces as necessary (ie, hold object up on table against gravity)
- When normal force = 0, objects no longer touch

→ Tension: a pull from a bar, rope, etc

- Acts like a force transfer between 2 objects



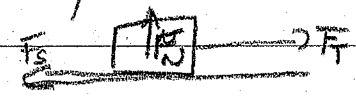
  - + Also can be given or adjust
  - + Can change direction over a pulley, etc due to forces w/pulley.



- Usually assume rope/bar is light + inextensible (no stretching)

→ Friction: force between two contacting objects parallel to the plane of contact

- Static friction: between 2 objects w/ no relative motion.
  + Acts against other forces to prevent relative motion up to max magnitude



$$|\vec{F}_s| \leq \mu_s |\vec{F}_N|$$

where μ_s = coefficient of static friction

- Kinetic friction: when objects have relative motion
  + Always directed opposite relative velocity + with magnitude $|\vec{F}_k| = \mu_k |\vec{F}_N|$ where μ_k = coefficient of kinetic friction

→ Air Resistance / Drag: friction-like force from moving through a medium (contact w/ molecules)

- Think of the drag force as a Taylor series in the relative velocity (directed against it)

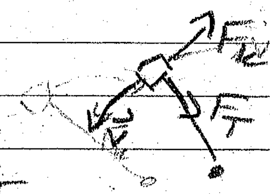
$$\vec{F} = \underbrace{-\mu \vec{v}/|\vec{v}|}_{\text{"constant"}} \underbrace{- \gamma \vec{v}}_{\text{linear}} \underbrace{- \zeta |\vec{v}| \vec{v}}_{\text{quadratic}} + \ldots$$

- For solids in contact, the constant term is most important (friction). But it's very small for fluids or air
- At low speeds, the drag is linear
  + $\vec{F} \doteq -\gamma \vec{v}$ γ
  + For sphere of radius R, $\gamma = 6\pi \eta R$, where η = viscosity of fluid

- At higher speeds, drag becomes quadratic
  + $\vec{F} = -\zeta |\vec{v}| \vec{v}$ with $\zeta \propto \rho A$, ρ = fluid density

and A = area of object. + As a result, drag is linear for Reynolds number $Re \equiv \rho v R/\eta \ll 1$ and quadratic for $Re \gg 1$.

+ An example: An object is constrained to move in a horizontal circle of radius R and initial angular velocity ω_0. With constant magnitude kinetic friction $\vec{F}$, what is tension as a function of time?



- Friction is tangential, tension radial / centripetal
  $m\dot{\omega}R = -F_k$, $F_T = m\omega^2 R$
  + With friction constant, $\omega = \omega_0 - (F_k/mR)t$, ...

- What if we replace friction with linear drag?
  $m\dot{\omega}R = -\gamma\omega R \Rightarrow \omega = \omega_0 \exp[-\gamma t/m]$

- Gravity: Universally attractive force between 2 objects. It's one of the fundamental forces

- Uniform gravity

  - Near the surface of Earth (or other large object like a planet), $\vec{F}_G = m\vec{g}$ on object of mass m

    + The gravitational mass is the same as the (inertial) mass experimentally. This equivalence principle is key to GR. Under tests

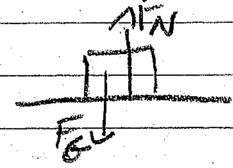
    + Setting $\hat{k}$ = upward, $\vec{g} = -g\hat{k}$. For earth, $g \approx 9.8\ m/s^2$

    + Potential energy is therefore $V = mgz = -m\vec{g}\cdot\vec{r}$

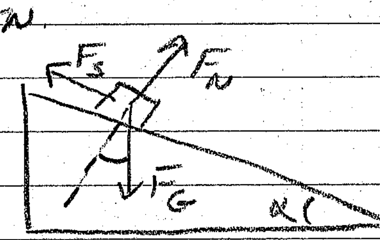
    + Using this formula approximates the earth as flat.

- Gravity often determines normal force

  + Gravity pulls the object down, but normal force keeps it on the surface

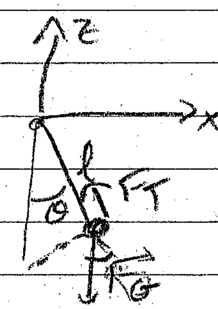


  + This also controls friction, since magnitudes F_s, F_k are controlled by F_N.



  + Ex. Block mass m sits on a wedge of angle α w/ coeff. static friction μ_s.
    For which α does the block slide?
    The block can't move $\perp$ the surface, so normal components cancel $F_N - mg\cos\alpha = 0$
    When the block is at rest, friction cancels gravity $F_s - mg\sin\alpha = 0 \Rightarrow \mu_s mg \cos\alpha$
    $\Rightarrow F_s = mg\sin\alpha \leq \mu_s mg\cos\alpha$.
    $\Rightarrow$ Box at rest for $\tan\alpha \leq \mu_s$

- The Pendulum: Mass hanging from a light rod (or string/wire) that moves on a circle (or possibly sphere)



  + Consider pendulum of mass m + length l.
    If θ = angle from $-z$ axis,
    $ml\dot\theta^2 = F_T - mg\cos\theta$ a centripetal, determines F_T
    $ml\ddot\theta = -mg\sin\theta$ a tangential
    From discussion of circular motion.
    Not generally easy to solve!

  + From energy perspective, $T = \frac{1}{2} m l^2 \dot\theta^2$ (see circular motion again) and
    $V = -m\vec{g}\cdot\vec{r} = -mgl\cos\theta$.

  + For total $E > mgl$, pendulum makes complete circles

+ For $E = -mgl\cos\theta_0$, $-\theta_0 \le \theta \le \theta_0$ (turning points)
Time from $\theta = 0$ to θ' is given by energy arguments
(take $m \to ml^2$ in previous formula by comparison)

$$t = \sqrt{l/2g} \int_0^{\theta} d\theta' / \sqrt{\cos\theta' - \cos\theta_0}$$ an Elliptic integral

The period is 4x time from 0 to θ_0.

+ When $\theta_0 \ll 1$, the elliptic integral simplifies
b/c $\cos\theta' \simeq 1 - \theta'^2/2$, $\cos\theta_0 \simeq 1 - \theta_0^2/2$, so

$$t = \sqrt{l/g} \int_0^{\theta} \frac{d\theta'}{\sqrt{\theta_0^2 - \theta'^2}} \quad \text{with } \theta' = \theta_0 \sin u,$$

$$= \sqrt{l/g} \int_0^{\sin^{-1}(\theta/\theta_0)} du = \sqrt{l/g}\, \sin^{-1}(\theta/\theta_0)$$

This is simple harmonic motion w/ frequency $\omega = \sqrt{g/l}$
You can get this from the tangential EOM w/ $\sin\theta \simeq \theta$.

+ Note that you can get the tangential EOM from energy conservation

$$dE/dt = l\dot{\theta}\,(ml\ddot{\theta} + mg\sin\theta) = 0$$

This is an important "trick" that will keep coming up.

- Ballistic Motion for the FAN (flat, airless, non-rotating) Earth

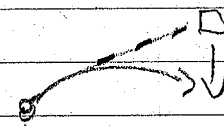
+ By conservation of momentum, motion is in a plane (force is only vertical). Assume it's in the (xz) plane with initial velocity $\vec{v}_0$ at angle α from the horizontal

+ The components of motion are

$$x = x_0 + v_0\cos\alpha\, t, \quad z = z_0 + v_0 \sin\alpha\, t - \tfrac{1}{2}gt^2$$

since acceleration is constant. This allows you to find horizontal range (change in x) as z reaches a particular height (like coming back to $z=0$)
This motion is parabolic

+ How far the object moves vertically is most easily determined w/ energy conservation. Energy conservation also means motion upward is symmetric w/ motion downward



+ Ex. An object falls from a height $z = h$. We want to throw a ball from $z = 0$ at a distance x away to strike the falling object. At what angle do we throw the ball? We see that both ball + falling object have the same acceleration. If we throw the ball when the object starts falling, we need to aim at the initial position of the object! As long as the ball travels far enough horizontally before the object hits the ground, they will strike.

• Projectile Motion with Air Resistance

+ Consider an object dropping in gravity with air resistance $\vec{F} = -\lambda |\vec{v}|^{n-1} \vec{v}$. As it drops, this points up. Total force $= 0$ at speed

$$v_T = (mg/\lambda)^{1/n} = \text{terminal velocity}$$

+ Suppose the drag is linear $\vec{F} = -\gamma \vec{v}$. Then the equation of motion $\dot{\vec{v}} = -(\gamma/m)\vec{v} + \vec{g}$. The general
Start by solving homogeneous eqn $\dot{\vec{v}} + (\gamma/m)\vec{v} = 0$
$\Rightarrow \vec{v} = \vec{v}_0 \exp(-\gamma t/m)$. Instead define $\vec{v}(t) = \vec{f}(t)\exp(-\gamma t/m)$
Then

$$\dot{\vec{f}}\, e^{-\gamma t/m} = \vec{g} \Rightarrow \vec{f} = \frac{m\vec{g}}{\gamma} e^{+\gamma t/m} + \vec{f}_0$$

If initial velocity is $\vec{v}_0$,

$$\vec{v} = m\vec{g}/\gamma + \vec{f}_0 \exp(-\gamma t/m) = \frac{m\vec{g}}{\gamma} + \left(\vec{v}_0 - \frac{m\vec{g}}{\gamma}\right) e^{-\gamma t/m}$$

Suppose $\vec{v}_0$ is purely vertical. As $t \to \infty$,
$\vec{v} \to m\vec{g}/\gamma$. That's the (terminal velocity) as above.

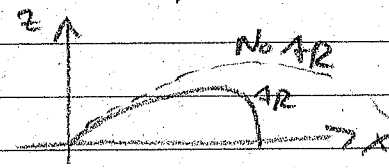
+ Suppose the particle is initially at the origin. Integrating velocity tells us

$$\vec{r} = \frac{m\vec{g}}{\gamma}t + \frac{m}{\gamma}\left(\vec{v}_0 - \frac{m\vec{g}}{\gamma}\right)\left(1-e^{-\gamma t/m}\right)$$

We can break this into components and solve for t in terms of x to get (setting $y=0$ always)

$$z = \frac{v_{0,z} + mg/\gamma}{v_{0,x}}x + \frac{m^2 g}{\gamma^2}\ln\left(1 - \gamma x/m v_{0,x}\right)$$

Path is not parabolic, and range from $z=0$ back to $z=0$ is transcendental (no closed form)



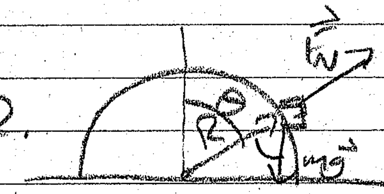
+ Think about different limits. At short times, expand the exponential. Same with $\gamma \to 0$ — note that terms with $1/\gamma$ must cancel.
At large times, horizontal position becomes nearly constant, so it's dropping almost vertically.

+ Quadratic air resistance can be solved in 1D (see text) but requires computers in 2D. The EOM

$$m\ddot{\vec{r}} = m\vec{g} - c|\vec{v}|\vec{v} \quad \text{to} \quad \ddot{x} = -c/m\sqrt{\dot{x}^2+\dot{z}^2}\,\dot{x}$$
$$\ddot{z} = -g - c/m\sqrt{\dot{x}^2+\dot{z}^2}\,\dot{z}$$

Coupled and nonlinear!

• Another example: An object slides down a frictionless sphere of radius R. If it starts at the top, at rest, where does it fall off? The 2nd law in $\hat{r}$ direction is (circular motion)



$$-mv^2/R = -mg\cos\theta + F_N$$

Energy conservation is

$$E = mgR = \tfrac{1}{2}mv^2 + mgR\cos\theta$$

Solving,

$$F_N = mg\cos\theta - 2(mg - mg\cos\theta) = mg(3\cos\theta - 2)$$

Object falls off sphere at $F_N = 0$, or $\cos\theta = 2/3$.

- Inverse Square Law

  - Newton's Law of Universal Gravitation
    - The force between objects of masses m_1 + m_2 & positions $\vec{r}_1$ and $\vec{r}_2$ is

$$\vec{F}_G = -G\frac{m_1 m_2}{|\vec{r}_1-\vec{r}_2|^3}(\vec{r}_1-\vec{r}_2) \quad \leftarrow \text{force on 1 by 2}$$

where G = Newton's constant $= 6.67\times 10^{-11}\ m^3/kg\,s^2$

    - We will consider the case where $m_2 = M \ggg m_1 = m$, so #2 basically doesn't move. Set $\vec{r}_2 = 0$. We therefore use

$$\vec{F}_G = -GMm\,\hat{r}/r^2 \quad (\vec{r}=\vec{r}_1)$$

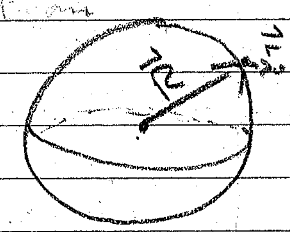
for the force on #1.

    - This is a conservative force with potential energy

$$V = -GMm/r$$

    - Can treat large spherical objects like point particles. See why later

  - Flat Earth approximation
    - We've previously approximated Earth as flat. That means the ~~displacement from~~ displacement $\vec{r}$ from an origin on the ground is small compared to the Earth's radius R



    - The potential energy is $V = -GMm/|\vec{R}+\vec{r}|$
This has approximation

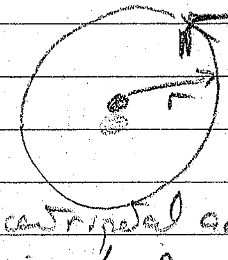
$$V = -GMm\left[(\vec{R}+\vec{r})^2\right]^{-1/2} = -GMm\left[R^2+r^2+2\vec{R}\cdot\vec{r}\right]^{-1/2}$$

$$\approx -\frac{GMm}{R}\left(1+\frac{2\vec{R}\cdot\vec{r}}{R^2}\right)^{-1/2} = -\frac{GMm}{R} + m\left(\frac{GM}{R^2}\hat{R}\right)\cdot\vec{r}$$

    - This is a constant (that we usually ignore) and a term $V = -m\vec{g}\cdot\vec{r}$ as usual with $\vec{g} = -GM\hat{R}/R^2$

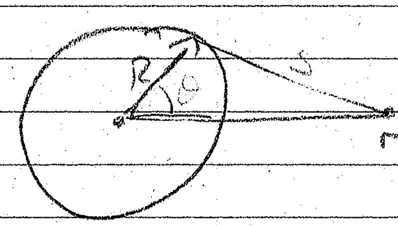
- Escape velocity
  - To escape the Earth's gravity (or another planet) means you need to reach distance $r \to \infty$
  - As $r \to \infty$, $V \to 0$, so the energy needed to escape is $E \geq 0$. When $E = 0$ the object coasts out to $r \to \infty$
  - This means that the initial velocity at Earth's surface comes from $E = \frac{1}{2}mv^2 - GMm/R \geq 0 \Rightarrow v \geq \sqrt{2GM/R} \equiv v_{esc}$
  - For initial $E < 0$, you go out to max distance + fall back

- Circular Orbits



  - Consider a large object (planet, star) at the origin w/ smaller object moving around it — orbit
  - For a circular orbit, gravitational force gives centripetal acc.
    $\Rightarrow$ speed in orbit determined by orbit size/radius
    $$mv^2/r = GMm/r^2 \Rightarrow v = \sqrt{GM/r}$$

- Gravity of a Spherical Shell
  - Why can we use the gravitational potential energy of 2 point particles if 1 object is very large?
  - Newton's law of gravitation is linear, so the potential energy of particle #1 in the presence of multiple objects is the sum of the potentials of #1 in the presence of each — Not true for GR, for ex.
  - Consider a uniform spherical shell of mass M. What is potential energy of mass m a distance r from the center? Shell has radius R.



  - We'll give the proof later: $V = \begin{cases} -GMm/r & \text{for } r \geq R \\ -GMm/R & \text{for } r < R \end{cases}$

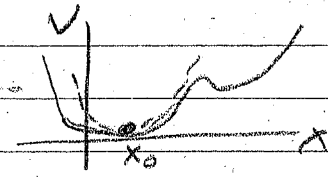
    $\Rightarrow$ zero force inside, force like point mass outside
    $\Rightarrow$ For any spherically symmetric mass $M(r)$, $\vec{F} = -GmM(r)/r^2\,\hat{r}$ where $M(r)$ = mass inside sphere of radius r

Harmonic Oscillators: system with linear restoring force. In 1D, $F = -kx$, $V = \frac{1}{2}kx^2$

- Physical Importance

  - Almost anything is a harmonic oscillator
    - Equilibrium is where an object can stay at rest. Force must be zero
    - Taylor expand potential around equilibrium x_0

      $V(x) \approx V(x_0) + (x-x_0)\, dV/dx|_{x_0} + \frac{1}{2}(x-x_0)^2\, d^2V/dx^2|_{x_0} + \ldots$



    - Equilibrium means $dV/dx = 0$
      Potential is approx quadratic $\Rightarrow$ Harmonic oscillator with $k = d^2V/dx^2|_{x_0}$

    - Stable equilibrium $d^2V/dx^2 > 0$
      Unstable equilibrium $d^2V/dx^2 < 0$. (top of hill)

    - Ex Pendulum: equilibrium is at $\theta = 0$ (tangential force vanishes). Potential energy is

      $V(\theta) = -mgl\cos\theta = -mgl + \frac{1}{2}mgl\theta^2 + \ldots$

  - Exactly solvable (pretty rare)

    - The EOM is $m\ddot{x} + kx = 0$
      w/ init. conditions $x(0) = x_0$, $\dot{x}(0) = v_0$
    - Solution is

      angular frequency

      $x(t) = x_0\cos(\omega_0 t) + v_0/\omega_0 \sin(\omega_0 t)$, $\omega_0 = \sqrt{k/m}$

      angle in cosine is phase $\rightarrow$ $= \sqrt{x_0^2 + (v_0/\omega_0)^2}\cos(\omega_0 t + \phi)$

      where $\tan\phi = v_0/\omega_0 x_0$ when $k > 0$

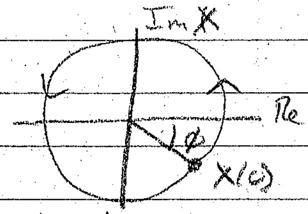
    - Alternately guess an exponential $x = Ae^{\beta t}$

      $\Rightarrow -m\beta^2 + k = 0 \Rightarrow \beta = \pm i\omega_0$

      Real solution $x(t) = (Ae^{i\omega_0 t} + A^* e^{-i\omega_0 t})/2$

This is as above w/ $A = \sqrt{x_0^2 + (v_0/\omega_0)^2}\, e^{-i\phi}/2$

+ The complex solution $x = Ae^{i\omega_0 t}$ shows why we use ω for frequency – it is angular velocity in the complex plane. Real sol'n is projection on real axis



+ The method of guessing an exponential solution also works near unstable equilibrium for $k < 0$.

— Damped Oscillators: Harmonic oscillator with a linear drag force $F = -\gamma \dot{x}$

- Time Dependence
  + Newton's 2nd law $m\ddot{x} + \gamma\dot{x} + kx = 0$ or
    $$\ddot{x} + 2\alpha\dot{x} + \omega_0^2 x = 0 \text{ for } \omega_0^2 = k/m,\ \alpha = \gamma/2m$$
  + α = damping coefficient, ω_0 = natural frequency
  + Guess $x = Ae^{\beta t} \Rightarrow \beta^2 + 2\alpha\beta + \omega_0^2 = 0$
    $\Rightarrow \beta = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$. Real or complex!
  + Each value of β gives one independent solution
- 3 behaviors
  + Underdamped $\omega_0 > \alpha$, β complex $= -\alpha \pm i\omega$
    The complex solution is $x = Ae^{-\alpha t} e^{\mp i\omega t}$
    with $\omega = \sqrt{\omega_0^2 - \alpha^2}$. A set by initial conditions
  + Overdamped $\alpha > \omega_0$. 2 real values of β.
    Displacement falls exponentially $x = Ae^{-\beta_+ t} + Be^{-\beta_- t}$
    At late times,
    $$x \approx Be^{-\beta_- t},\ \beta_- = \alpha - \sqrt{\alpha^2 - \omega_0^2}$$
  + Critically damped $\alpha = \omega_0$. Slight subtlety here due to double root.
    $$x = (A + Bt)\, e^{-\alpha t}$$

- Forced (or Driven) Oscillator
  - Suppose we have a damped oscillator that we pull with an additional periodic force $F(t) = F_0 \cos(\omega t)$
    + The oscillator eqn. is linear, so we can instead use $F(t) = F_0 e^{i\omega t}$ + add solutions. This lets us use exponential solutions

    + The EOM is $\ddot{x} + 2\alpha\dot{x} + \omega_0^2 x = (F_0/m) e^{i\omega t}$
    The general solution is a particular solution for this force plus the 2 homogeneous sol'ns of the damped oscillator. Those have integration constants used to match initial conditions

    + Guess a particular solution $x(t) = A e^{i\omega t}$
    Then $A(-\omega^2 + 2i\alpha\omega + \omega_0^2) = F_0/m$

    + For $A = |A| e^{-i\theta}$, $|A| = (F_0/m) / \sqrt{(\omega_0^2 - \omega^2)^2 + 4\alpha^2\omega^2}$
    and
    $$\tan\theta = 2\alpha\omega / \omega_0^2 - \omega^2 \quad \text{(Derive)}$$

  - Physics of solution
    + Homogeneous solutions fall off. Called transients. Late time behavior is particular solution

    + θ = phase lag. Force starts increasing, but then x does later. Can be totally out of phase.

    + Amplitude gets large at resonance near $\omega \approx \omega_0$

  - If the forcing is a different function, you can use Fourier series/transforms or other ODE techniques

- 2D or 3D oscillators: There is always a set of $\perp$ axes such that $V(\vec{r}) = \frac{1}{2}k_1 x^2 + \frac{1}{2}k_2 y^2 \left(+ \frac{1}{2}k_3 z^2\right)$ looks like independent oscillation in each direction.