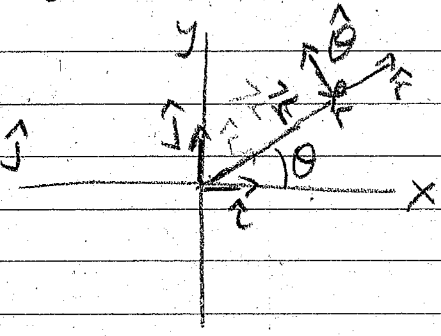
Central Forces + Orbits

⓪ Central Forces + Angular Momentum

- Central Forces in 3D
  - A central force is radial
    - + The force is directed along the dispacement from the origin + depends only on radius
      $$\vec{F} = F(r)\,\hat{r}$$ ← case we'll consider
    - + A central force between 2 objects depends on the displacement between them
      $$\vec{F} = F(|\vec{r}_1-\vec{r}_2|)\,(\vec{r}_1-\vec{r}_2)/|\vec{r}_1-\vec{r}_2|$$
    - + From now on we'll look at 1st case
  - Potential depends on radius only $V=V(r)$ (conservative)
    - + If you use spherical polar coords (maybe see later in term) $\vec{F} = -\vec{\nabla}V = -(\partial V/\partial r)\,\hat{r}$
    - + We can see this directly from $r=\sqrt{x^2+y^2+z^2}$
      $$\vec{\nabla}V(r) = \frac{dV}{dr}\frac{\partial r}{\partial x}\hat{\imath} + \frac{dV}{dr}\frac{\partial r}{\partial y}\hat{\jmath} + \frac{dV}{dr}\frac{\partial r}{\partial z}\hat{k}$$
      $$= \frac{dV}{dr}\left(\frac{x}{r}\hat{\imath} + \frac{y}{r}\hat{\jmath} + \frac{z}{r}\hat{k}\right) = \frac{dV}{dr}\hat{r}$$
    - + This is like a 1D force
  - Motion in a central force is confined to a plane (we'll prove soon). Suggests we can use plane polar coords.

- Plane Polar Coordinates: remember circular motion
  - Label points by distance r from origin + angle θ from x-axis
    - + Position is $\vec{r} = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath}$
      so $\hat{r} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$
    - + Tangential unit vector is $\perp$
      $$\hat{\theta} = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$$



• Cylindrical coords. have polar in (x,y) plane w/ $\hat{r}$, $\hat{\theta}$ unit vectors and $\hat{k}$ along z axis

• These unit vectors depend on position

$d\hat{r}/d\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} = \hat{\theta}$,

$d\hat{\theta}/d\theta = -\cos\theta\,\hat{i} - \sin\theta\,\hat{j} = -\hat{r}$

• Cylind

• Assuming position $r(t)$, $\theta(t)$ depend on time

+ Position is $\vec{r} = r\hat{r}$

+ Velocity is $\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}(d\hat{r}/d\theta)$

or $\vec{v} = \dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \equiv v_r\hat{r} + v_\theta\hat{\theta}$

+ Acceleration is

$$\vec{a} = \ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

+ Radius r = const. is circular motion → Compare!

• What are the terms in $\vec{a}$? There is motion along r and around θ. $-r\dot{\theta}^2\hat{r}$ is centripetal acceleration. $2\dot{r}\dot{\theta}\hat{\theta}$ is the Coriolis effect - moving both directions leads to swirling motion

- For a central force, 2^{nd} law is

$\ddot{r} - r\dot{\theta}^2 = F(r)/m \qquad r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$

$\Rightarrow \frac{d}{dt}(r^2\dot{\theta}) = 0 \Rightarrow r^2\dot{\theta} = \text{const}$

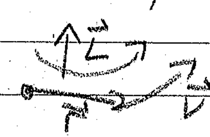
Looks like a conservation law

- Angular Momentum

• Define a quantity angular momentum $\vec{L}(t) = \vec{r}(t) \times \vec{p}(t)$

+ Sometimes it might be called $\vec{J}$

+ Represents tangential type motion



• Torque

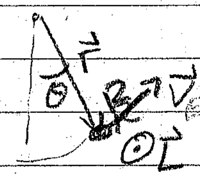
+ Time derivative is $d\vec{L}/dt = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}} = \vec{r} \times \dot{\vec{p}}$

b/c $\vec{p} = m\dot{\vec{r}} \Rightarrow d\vec{L}/dt = \vec{r} \times \vec{F} \equiv \vec{\tau}$

+ $\vec{\tau} \equiv \vec{r} \times \vec{F} \equiv$ torque. Sometimes called "moment of force" and maybe labeled $\vec{G}$, $\vec{\Gamma}$, or $\vec{N}$.

+ When $\vec{\tau} = 0$, angular momentum is conserved.
Later, we'll argue that angular momentum of multiple particles changes only due to an external torque

- Example: The pendulum



  + Velocity is $\vec{v} = R\dot{\theta}\hat{\theta}$,
    so $\vec{L} = mR^2\dot{\theta}\,\hat{k}$ out of the page
    (or into for negative $\dot{\theta}$)

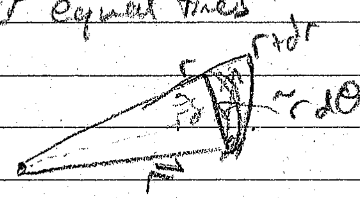
  + Torque is $\tau = \vec{r} \times (m\vec{g}) = -mRg\sin\theta$
    (into page for $\theta > 0$)
  + EOM is therefore $mR^2\ddot{\theta} = -mRg\sin\theta$
    $\Rightarrow \ddot{\theta} = -(g/R)\sin\theta \approx -(g/R)\theta$
    This is another way to get this equation

- Torque vanishes for a central force
  $\vec{\tau} = \vec{r} \times \vec{F} = |r| F(r) \hat{r} \times \hat{r} = 0$
  Angular momentum is conserved

- Motion in a central force is planar
  + $\vec{L}$ is conserved. Choose $\vec{L} = L\hat{k}$
    (choice of axes)
  + But $\vec{r} \cdot \vec{L} = \vec{r} \cdot (\vec{r} \times \vec{p}) = 0$ always.
    $\Rightarrow$ Object stays in (xy) plane.

  + In cylindrical coordinates,
    $\vec{L} = m\vec{r} \times \vec{v} = mr^2\dot{\theta}\,\hat{k}$
    Our conserved quantity is $l = r^2\dot{\theta} = |\vec{L}|/m$
    = ang. mom. per unit mass

- The area swept out by $\vec{r}$ is equal over equal times
  + For small times, the area swept is $\approx$ area of isoceles triangle w/ height $\approx r$ and base $\approx r\,d\theta$.
    $\Rightarrow dA = \frac{1}{2} r^2 d\theta$



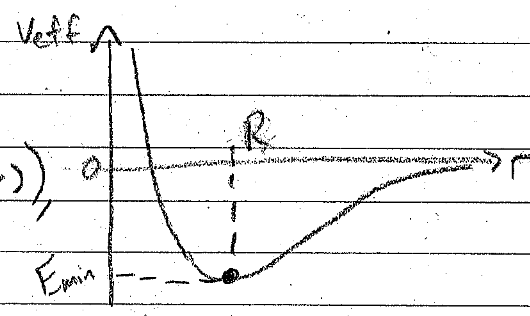
+ Area per time is $\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = l/2 = \text{constant}$
since ang. mom. is constant
+ This is Kepler's 2nd law. Originally observed for planets but true for any central force.

- The Effective Potential: looking like 1D system
  - As we saw before, 2nd law gives
    $$m(\ddot{r} - r\dot{\theta}^2) = F(r) \quad \text{and} \quad l \equiv r^2\dot{\theta} = \text{constant}$$
    + We can substitute for $\dot{\theta}$
    $$m\ddot{r} - ml^2/r^3 = F(r) = -dV/dr$$
    + This looks like a force with a centrifugal term
    $$m\ddot{r} = F(r) + ml^2/r^3 = -dV/dr + ml^2/r^3$$
    + This is conservative and comes from an effective potential energy $V_{eff} = V(r) + ml^2/2r^2$
    $$-dV_{eff}/dr = F(r) + ml^2/r^3$$

  - We can also see V_{eff} in the energy
    + Conserved energy is
    $$E = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 + V(r)$$
    $$= \tfrac{1}{2}m\dot{r}^2 + ml^2/2r^2 + V(r) = \tfrac{1}{2}m\dot{r}^2 + V_{eff}(r)$$

    + Again, this is an effectively 1D system with potential energy given by $V_{eff}(r)$.
    + The $ml^2/2r^2$ term is a barrier from reaching the origin due to angular momentum

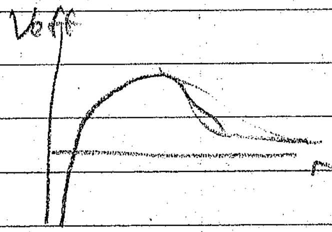
  - Orbits when $V(r) \to 0$ as $r \to \infty$
    + If $E > 0$, the particle can escape to infinity
    + If $E = E_{min} \equiv \min(V_{eff}(r))$, this means $\dot{r} = 0$.
    So $r = R$ = constant but
    $\dot{\theta} = l/R^2 > 0 \Rightarrow$ this is circular motion



The radius of the orbit R is at the minimum, $\frac{dV_{eff}}{dr}(R) = 0$

+ For $E_{min} < E < 0$, the radius oscillates between 2 points called <u>apsides</u>. The closest point (smallest r) is the <u>periapsis</u> (or pericenter); farthest is <u>apoapsis</u> (apocenter). Due to angular momentum, this is an orbit in the plane.

+ There is a circular orbit whenever $dV_{eff}/dr = 0$. These are stable (oscillation around orbit) at a minimum of V_{eff}. But you can have a case like this w/ unstable orbits



- Orbits in Gravity
  - Solution for motion in gravity
    - Set-up: as usual, we have an object of mass m moving around an object of mass $M \gg m$
      + Since M is so large, we approximate that it doesn't move. We locate it at the origin
      + Force + potential energy are
        $\vec{F} = -(mk/r^2)\hat{r}$, $V = -mk/r$ where $k = GM$ (saves writing)
      + The Coulomb force between charges is the same with $k = -q_1 q_2 / 4\pi\epsilon_0 m$. This is a repulsive force when $k < 0$.
    - Since $\dot{\theta} = l/r^2$, we can solve all motion if we get $r(t)$.
      + From energy conservation, remember we know
        $$t(r) = \int dt/\dot{r} = \int dr \sqrt{\frac{m/2}{E - V_{eff}(r)}} \text{ + inverse}$$
      + For gravity, $V_{eff} = -mk/r + ml^2/2r^2$
        The integral gives a transcendental equation — not helpful
      + for finding $r(t)$
    - What we really want is the shape $r(\theta)$.
      + θ is a monotonically increasing function of t. So it can be a substitute "time" variable.

+ We want $r' = \frac{dr}{d\theta}$. By chain rule, $\dot{r} = \dot{\theta} r' = l r'/r^2$.
Simplify more by taking $u = 1/r$. Then
$$\dot{r} = -l u'$$
+ You can get $\ddot{r}$ and use the 2nd law (see Tong reading)
Instead, let's look at...

- Energy considerations
  + With these variables, the energy is
$$E = \tfrac{1}{2} m \dot{r}^2 + m l^2/2r^2 - mk/r = \frac{ml^2}{2}\left[(u')^2 + u^2 - \left(2k/l^2\right)u\right]$$
  + If we complete the square, $\left(E + \frac{mk^2}{2l^2}\right) = \frac{ml^2}{2}\left[(u')^2 + \left(u - \frac{k}{l^2}\right)^2\right]$
  + This looks like the energy of a harmonic oscillator with shifted origin and "frequency" $= 1$
  Therefore, $u = k/l^2 + A\cos\theta + B\sin\theta$
  + Let's set $\theta = 0$ as the smallest radius r.
  Then
$$\frac{1}{r} = u = \frac{k}{l^2}\left(1 + e\cos\theta\right) \quad e = \text{integration const}$$
  + or
$$1/r = |k|/l^2\left(e\cos\theta - 1\right) \quad \text{for } k<0 \text{ (repulsive force)}$$
  Note that the integration const $e > 1$ in this case and $|\cos\theta| \geq 1/e$.
  + Plugging back in, $E = \frac{mk^2}{2l^2}\left(e^2 - 1\right)$ ← check units yourself!
  This is $E \geq 0$ for $e \geq 1$ (object escapes)
  and $E < 0$ for $e < 1$ (bound orbit).
  For a bound orbit, we define the length $l^2/k(1-e^2) \equiv a$
  so $E = -mk/2a$

- Interpreting the shape.
  - Start with a bound orbit, $e < 1$.
    + Write our solution as $r(1 + e\cos\theta) = a(1-e^2)$

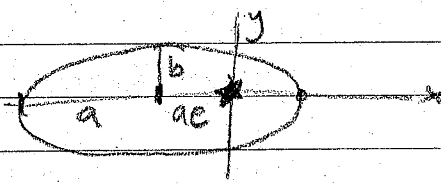
or $r = a(1-e^2) - ex$ b/c $x = r\cos\theta$

+ Square to get $x^2+y^2 = a^2(1-e^2)^2 - 2ae(1-e^2)x + e^2x^2$
  Gather terms w/ x and complete squares

$$(1-e^2)(x+ae)^2 + y^2 = a^2(1-e^2)$$

This is

(4) $$(x+ae)^2/a^2 + y^2/b^2 = 1 \quad \text{where } b = a\sqrt{1-e^2} < a$$



+ This is the formula of an ellipse with center at $x = -ae$ and one focus at the origin
  a = semi-major axis, b = semi-minor axis
  e = eccentricity of ellipse
+ Notice that semi-major axis determines energy (or vice versa). Then ang. momentum determines eccentricity

+ A circular orbit is $e = 0$.

• Escape velocity is the boundary between bound and escaping orbits, $e = 1$.

$$r = l^2/k - x \Rightarrow y^2 = (l^2/k)^2 - 2(l^2/k)x$$

Parabolic!

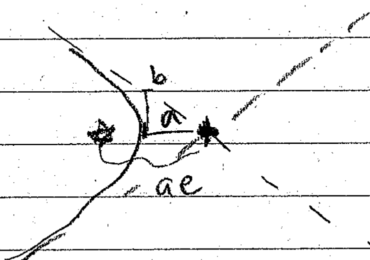
• Escaping orbits with $e > 1$ for attractive potentials

+ For $e > 1$, $r \to \infty$ when $\cos\theta = -1/e$

$$r = (l^2/k)/(1+e\cos\theta)$$

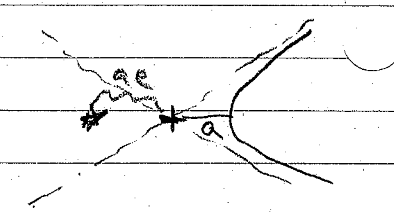
+ We can define $a = l^2/k(e^2-1)$ and $b = a\sqrt{e^2-1}$.
  Then similar steps give

$$\frac{(x-ae)^2}{a^2} - \frac{y^2}{b^2} = 1$$



+ This is a hyperbola with center at $x = +ae$ & focus at the origin. The 2 asymptotes have slope $\pm b/a$.

- Repulsive potentials $e > 1$
  - + Again, define $a = l^2/|k|(e^2-1)$.
    Then $r(e\cos\theta - 1) = a(e^2-1)$
    $\Rightarrow (x-ae)^2/a^2 - y^2/b^2 = 1$
    again!
  - + This is the "far" branch of the same hyperbola!



— Kepler's Laws of Planetary Motion

- 1st Law: Each planet moves on an ellipse with the sun at one focus. Check!

- 2nd Law: The line from the sun to the planet (or planet to satellite, etc) sweeps out equal areas in equal times.
  We also already proved this from angular momentum conservation (no gravity!)

- 3rd Law: The square of the orbit's period is proportional to the cube of the semimajor axis

  - + Consider Kepler's 2nd law $dA/dt = l/2 =$ constant
  - + So for an ellipse, the period $T = 2A/l$
    where $A =$ ellipse area $= \pi ab$

  - + To simplify: $b = a\sqrt{1-e^2}$ and $l = \sqrt{ak(1-e^2)}$
    Then
    $$T = 2\pi a^2\sqrt{1-e^2} \Big/ \sqrt{ak(1-e^2)} = 2\pi\sqrt{a^3/k}$$
    $\Rightarrow T^2 = 4\pi^2 a^3/GM$
  - + This proportionality holds for all objects orbiting the same larger object, ie, planets around the sun or satellites around earth, but the proportionality constant changes when the central object changes.

- In astro, earth's semi-major axis = 1 AU = astronomical unit. Often used distance in AU, and time in years.