Accelerating Reference Frames

What if an inertial reference frame is inconvenient?

Accelerating Frames + Fictitious Forces

- Consider the motion of a particle as measured by an inertial frame S and in an accelerating frame S'. Position in the 2 frames is $\vec{r}$ or $\vec{r}'$ respectively.
  - We will show coming up that the acceleration in the 2 frames is related by

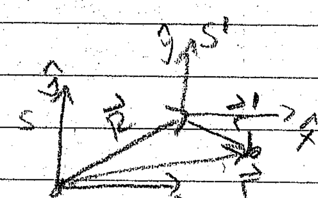
$$\left.\frac{d^2\vec{r}}{dt^2}\right|_S = \left.\frac{d^2\vec{r}'}{dt^2}\right|_{S'} + \Delta\vec{a}$$

  - What is Newton's 2nd law?
    + Of course, in S, $\vec{F} = m d^2\vec{r}/dt^2$
    + Translates in S' to

$$m\frac{d^2\vec{r}'}{dt^2} = \vec{F} - m\Delta\vec{a}$$

    + To say that the 2nd law holds in frame S', we have to identify $-m\Delta\vec{a}$ as a force, even though it is not caused by a physical object or force. Call it a <u>fictitious force</u>.

- Acceleration of the origin



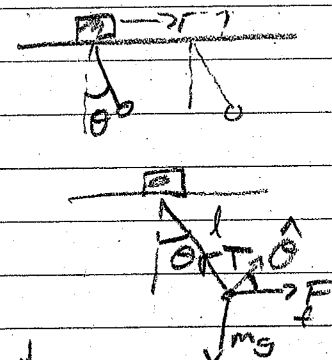
  - Our inertial frame S has axes $\hat{i}, \hat{j}, \hat{k}$, and the accelerating frame S' has fixed axes $\hat{x}, \hat{y}, \hat{z}$ and origin at $\vec{R}(t)$ with respect to frame S.
    + The positions of an object measured from S + S' satisfy

$$\underset{S}{\vec{r}} = \underset{S'}{\vec{r}'} + \vec{R}$$

    + Then the accelerations are related by

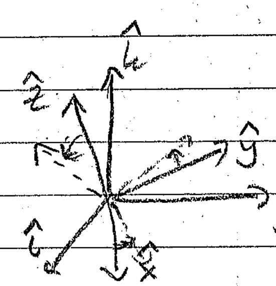
$$\frac{d^2\vec{r}}{dt^2} = \frac{d^2\vec{r}'}{dt^2} + \frac{d^2\vec{R}}{dt^2}$$

- Objects experience fictitious force $\vec{F}_f = -m\, d^2\vec{R}/dt^2$ from the point of view of frame S'
  - + This is the case for the frame inside an elevator, an automobile, etc.
  - + This looks a lot like the force law for gravity (uniform) (mass × const acc.). In GR, gravity is fictitious!



- Example: A pendulum hangs from a support that moves with position
  $$x(t) = A\cos(\omega t)$$
  - + In accelerating frame, let θ = angle from instantaneous vertical under support
  - + The forces in the $\hat{\theta}$ direction are gravity + fictitious
  $$m l\ddot{\theta} = -mg\sin\theta + F_f\cos\theta.$$
  - + The fictitious force is
  $$F_f = -m\ddot{x} = mA\omega^2\cos(\omega t)$$
  so in small θ limit,
  $$ml\ddot{\theta} + mg\theta = mA\omega^2\cos(\omega t)$$
  Pendulum is a forced oscillator.

— Rotating Axes:



- Take an inertial frame S w/ unit vectors $\hat{i}, \hat{j}, \hat{k}$ along fixed axes. A frame S' has the same origin but rotating axes with rotating unit vectors $\hat{x}, \hat{y}, \hat{z}$
  - + We will always use $\hat{i}, \hat{j}, \hat{k}$ to mean inertial, time-independent unit vectors. Meanwhile, we'll use $\hat{x}, \hat{y}, \hat{z}$ for rotating axes ← avoids keeping track of primes
  - + The rotation of the axes is described by an angular velocity vector $\vec{\omega}$. + Suppose there is a particle with fixed position in S', meaning $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ w/ x, y, z = constant.

This particle is undergoing (instantaneously) circular motion in the S frame, so we know $\frac{d\vec{r}}{dt}\Big|_S = \vec{\omega}\times\vec{r}$ in inertial frame S.

+ How does this work w/ components constant in S'? Because they are rotating, $\hat{x}, \hat{y}, \hat{z}$ depend on time vs $\hat{i}, \hat{j}, \hat{k}$. The above eqn. for $\vec{r}$ only works if

$$\frac{d\hat{x}}{dt}\Big|_S = \vec{\omega}\times\hat{x}, \quad \frac{d\hat{y}}{dt}\Big|_S = \vec{\omega}\times\hat{y}, \quad \frac{d\hat{z}}{dt}\Big|_S = \vec{\omega}\times\hat{z}$$

+ This is like unit vectors in polar coordinates.

- Now suppose the particle is moving around generally
  + The position is the same vector in both frames but written with different components

$$\vec{r} = r_1\hat{i} + r_2\hat{j} + r_3\hat{k} = x\hat{x} + y\hat{y} + z\hat{z}$$

  + If we are in S frame, we see changes in x and $\hat{x}$, etc. But if we rotate with S', $\hat{x}, \hat{y}, \hat{z}$ look constant. So we think

$$\frac{d\vec{r}}{dt}\Big|_{S'} = \left(\frac{dx}{dt}\right)\hat{x} + \left(\frac{dy}{dt}\right)\hat{y} + \left(\frac{dz}{dt}\right)\hat{z}$$

  + Therefore, comparing the 2 frames using product rule

$$\frac{d\vec{r}}{dt}\Big|_S = \frac{d\vec{r}}{dt}\Big|_{S'} + \vec{\omega}\times\vec{r}$$

  + The same logic applies to any vector, like momentum, whatever

$$\frac{d\vec{b}}{dt}\Big|_S = \frac{d\vec{b}}{dt}\Big|_{S'} + \vec{\omega}\times\vec{r}$$

  + Note that $\frac{d\vec{\omega}}{dt}\Big|_S = \frac{d\vec{\omega}}{dt}\Big|_{S'} = \dot{\vec{\omega}}$

In the future, a dot is time derivative w.r.t. S'

- Apply the general formula to velocity using product rule

$$\frac{d^2\vec{r}}{dt^2}\Big|_S = \ddot{\vec{r}} + 2\vec{\omega}\times\dot{\vec{r}} + \dot{\vec{\omega}}\times\vec{r} + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

So Newton's 2nd law is

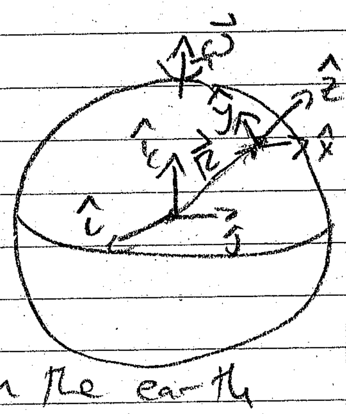
$$m\ddot{\vec{r}} = \vec{F} - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) - 2m\vec{\omega}\times\dot{\vec{r}} - m\dot{\vec{\omega}}\times\vec{r}$$

where $\vec{F}$ = physical force and the rest are fictitious

+ $-m\vec{\omega}\times(\vec{\omega}\times\vec{r}) = -m(\vec{\omega}\cdot\vec{r})\vec{\omega} + m\omega^2\vec{r}$ = centrifugal force that points away from axis of rotation w/ magnitude $m\omega^2 d$ where d = distance from axis

+ $-2m\vec{\omega}\times\dot{\vec{r}}$ = Coriolis force. If $\vec{\omega}$ points up, it causes moving particles to bend clockwise
+ $-m\dot{\vec{\omega}}\times\vec{r}$ = Euler force (aka transverse force aka azimuthal force) due to change in rotation

- At the Earth's Surface



• A typical reference frame in daily life has a moving origin on the surface of the earth and axes that rotate with the earth

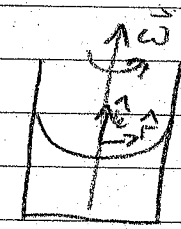
• Start with an inertial frame S w/ origin at the center of the earth.
  + Choose fixed $\hat{i}, \hat{j}, \hat{k}$ unit vectors, $\hat{k}$ pointing north
  + The earth rotates with $\vec{\omega} = \omega\hat{k}$, $\omega = 2\pi/\text{day}$
  + Our origin on the surface of the earth therefore has
$$\left.\frac{d\vec{R}}{dt}\right|_S = \vec{\omega}\times\vec{R}, \quad \left.\frac{d^2\vec{R}}{dt^2}\right|_S = \dot{\vec{\omega}}\times\vec{R} + \vec{\omega}\times(\vec{\omega}\times\vec{R})$$

• We work in the rotating frame S' w/ origin at $\vec{R}$ in S
  + Choose unit vectors/axes with
$$\hat{x} = \hat{\phi} = \text{east}, \quad \hat{y} = -\hat{\theta} = \text{north}, \quad \hat{z} = \hat{r} = \text{up}$$
  + The fictitious forces an object moving w/ position $\vec{r}$ relative to the S' origin feels are
$$-m\dot{\vec{\omega}}\times(\vec{R}+\vec{r}) - m\vec{\omega}\times(\vec{\omega}\times(\vec{R}+\vec{r})) - 2m\vec{\omega}\times\dot{\vec{r}}$$

+ For earth $\dot{\vec{\omega}}$ is small, so we ignore Euler force
+ We can also approximate centrifugal force as
$-m\vec{\omega}\times(\vec{\omega}\times\vec{R})$ for motion near surface $|\vec{r}| \ll |\vec{R}|$

Centrifugal Force

- The rotating bucket



• Suppose you have a bucket with water in it rotating around the axis $\vec{\omega} = \omega\hat{z}$. ~~What is~~
+ What is the shape of the water's surface?
+ The net force (physical + fictitious) must be $\perp$ to the surface, or else the molecules would move

• The forces are $\vec{F}_G = -mg\hat{z}$, $\vec{F}_{cent} = m\omega^2 r\hat{r}$ in cylindrical coordinates (check the cross products)
+ We can write this in terms of a potential
$$V = mgz - \tfrac{1}{2}m\omega^2 r^2$$
+ If $\vec{F} \perp$ surface, that means surface is an equipotential, ie, $V = \text{const}$ along surface
+ Therefore,
$$z = \frac{\omega^2 r^2}{2g} + \text{const} \quad \text{for surface of water.}$$

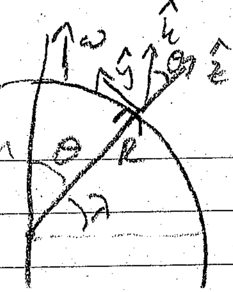
- Apparent Gravity of the Earth

• The force on a stationary object at earth's surface is $\vec{F}_G + \vec{F}_{cent}$

+ This is $m\left[-g\hat{z} - \vec{\omega}\times(\vec{\omega}\times\vec{R})\right] \equiv -m\vec{g}_{eff}$
where $\vec{g}_{eff}$ is the *effective* or *apparent* gravitational acceleration

+ To understand the size of centrifugal force
$|\vec{g}_{eff}| = g$ at north pole and $|\vec{g}_{eff}| = g - \omega^2 R$ at equator. The difference is
$34\ mm/s^2$ ~~or~~ few percent of g.

• The direction of apparent gravity changes with latitude (a little)



+ Latitude λ = angle from equator.

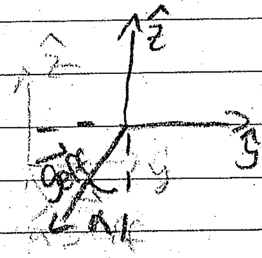
  $\lambda = \pi/2 - \theta$, where θ = polar angle

  so $\sin\theta = \cos\lambda$, $\cos\theta = +\sin\lambda$

+ In terms of rotating axes, $\hat{k} = \sin\theta\,\hat{y} + \cos\theta\,\hat{z}$,

  so

  $$\vec{\omega}\times(\vec{\omega}\times\vec{R}) = \omega^2 R\ \hat{k}\times(\sin\theta\,\hat{y}\times\hat{z}) = \omega^2 R\sin\theta\ \hat{k}\times\hat{x}$$
  $$= -\omega^2 R(\sin^2\theta\,\hat{z} - \sin\theta\cos\theta\,\hat{y})$$



+ $\Rightarrow \vec{g}_{eff} = -(g - \omega^2 R\cos^2\lambda)\hat{z} - \omega^2 R\sin\lambda\cos\lambda\,\hat{y}$

  which is at angle α from $\hat{z}$ axis

  with

  $$\alpha \approx \tan\alpha = \frac{\omega^2 R\sin\lambda\cos\lambda}{g - \omega^2 R\cos^2\lambda} \approx \frac{\omega^2 R\sin(2\lambda)}{2g}$$

  using small angle approx.

+ At maximum, $\alpha \approx 6$ arcmin $\approx 1/10$ deg $\approx 2\times10^{-3}$ rad

  So this is altogether small

+ Deflection points toward equator.

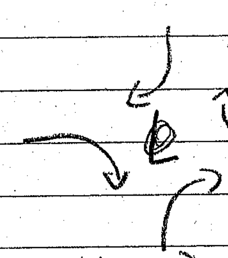
+ In principle, we could define $\hat{z}'$ opposite $\vec{g}_{eff}$ as "up". But the difference is so small we'll ignore it. You can also account for it by shifting λ by α.

• In reality, $\vec{g}_{eff}$ is also affected by local features like lakes or mountains. The value $g = 9.8\,m/s^2$ is a calculated value, not actually a measured one.

• Coriolis Effect

– Hurricanes

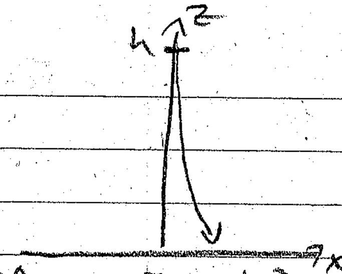
  ◦ Fluid, like the atmosphere, flows into a low pressure region



  ◦ Each stream bends clockwise (counter-) in northern (southern) hemisphere b/c $\vec{\omega}$ points partly up (down)

  ◦ This makes large storms swirl in the opposite sense

- Falling Objects



- Suppose you climb a tower + drop a ball vertically. Where does it land?
  + A weighted rope hangs straight down $\hat z$ axis. (Remember we can just shift latitude to account for apparent gravity)
  + We will find the ball lands a little east. Recall $\hat x$ = east, $\hat y$ = north

- Forces are gravity + Coriolis, so Newton's 2nd law is

$$m\ddot{\vec r} = m\vec g - 2m\vec\omega\times\dot{\vec r}$$

  + As before $\vec\omega = \omega\hat k = \omega(\cos\lambda\,\hat y + \sin\lambda\,\hat z)$
  + We have initial position $\vec r_0$ and initial velocity $= 0$ (when we drop it)

  If we integrate w.r.t. from 0 to t,

$$\dot{\vec r}(t) = \vec g t - 2\vec\omega\times[\vec r(t) - \vec r_0]$$

- To solve, we recognize $\omega = (2\pi/1\,\text{day})$ is small and approximate.
  + If we substitute back $\dot{\vec r}$,

$$\ddot{\vec r} = \vec g - 2\vec\omega\times[\vec g t - 2\vec\omega\times(\vec r - \vec r_0)]$$

  + The last term is proportional to ω^2, so we ignore it. Alternately, notice that $gt \sim 10\,m/s$ for a drop of $\sim 1s$ while $\omega r_0 \sim 10\,m/\text{day}$ for a similar drop. Comparatively negligible
  + Now we can integrate $\ddot{\vec r}$ twice

$$\vec r = \vec r_0 + \tfrac12\vec g t - \tfrac13(\vec\omega\times\vec g)t^3$$

  b/c $\vec g$ and $\vec\omega$ are constant.
  + We have $\vec g = -g\hat z$ and $\vec\omega\times\vec g = -\omega g\cos(\lambda)\,\hat x$

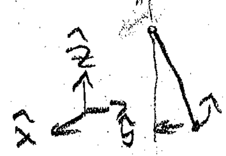
  With $\vec r_0 = h\hat z$, the components are

$$z = h - \tfrac12 g t^2 \Rightarrow \text{ball hits } z=0 \text{ at } t = \sqrt{2h/g}$$

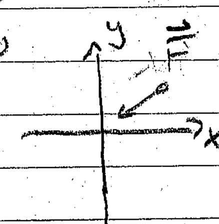
$$x = \tfrac13\omega g\cos(\lambda)\,t^3 \Rightarrow \text{ball reaches}$$

$$x = \tfrac{2\omega}{3}\cos(\lambda)\sqrt{2h^3/g} \text{ when it lands at } z=0.$$

- Comments
  + The deflection is always east
  + For $\lambda = 45°$ latitude, $h = 100\,m$, $x \approx 16\,mm$. Small!
  + You can add initial velocity, change $\vec r_0$, etc for a thrown ball, etc.



- Foucault's Pendulum
  - This is a pendulum that can swing in any direction but stays a distance l from the support point
    + In the small angle approximation, we know tension + gravity combine to make a pendulum into a harmonic oscillator in horizontal direction. So let's replace the pendulum with a harmonic oscillator of restoring force $\vec{F} = -k(x\hat{x} + y\hat{y})$



    + We also need the Coriolis force
      $\vec{F}_c = -2m\vec{\omega}\times\dot{\vec{r}}$ with $\vec{\omega} = \omega(\cos\lambda\,\hat{y} + \sin\lambda\,\hat{z})$
      $\Rightarrow \vec{F}_c = -2m\omega\left[-\cos\lambda\,\dot{x}\,\hat{z} + \sin\lambda\,\dot{x}\,\hat{y} - \sin\lambda\,\dot{y}\,\hat{x}\right]$
    The $\hat{z}$ component just tries to lift the oscillator off the floor, but it's not strong enough

  - Newton's 2nd law in horizontal directions has components
    $$\ddot{x} = -(k/m)x + 2\omega\sin(\lambda)\,\dot{y}$$
    $$\ddot{y} = -(k/m)y - 2\omega\sin(\lambda)\,\dot{x}$$
    + This is a coupled set of ODEs. To solve, define $u = x + iy$
      Then
      $$\ddot{u} + i2\omega\sin(\lambda)\,\dot{u} + (k/m)\,u = 0$$
    + That's like a complex harmonic oscillator equation, so try our guess $u = A\exp(i\beta t)$
      Then $-\beta^2 - 2\omega\sin(\lambda)\beta + (k/m) = 0$
      Solutions are $\beta = -\omega\sin(\lambda) \pm \sqrt{\omega^2\sin^2\lambda + k/m}$
      $\approx -\omega\sin(\lambda) \pm \sqrt{k/m}$

  - In other words,
    $$u = e^{-i\omega\sin(\lambda)t}\left[A\cos\left(\sqrt{\tfrac{k}{m}}\,t\right) + B\sin\left(\sqrt{\tfrac{k}{m}}\,t\right)\right]$$
    + For A real, $B = 0$, that is motion that starts out oscillating rapidly along the x axis + then rotates around (xy) plane w/ frequency $\omega\sin(\lambda)$.
    + More general choices of A + B give rotating elliptical motion
    + At Winnipeg's latitude, a full rotation is ~31 hours.
      But you can see a repeat every 15.5hr b/c oscillation along $+x$ and $-x$ axes look the same.