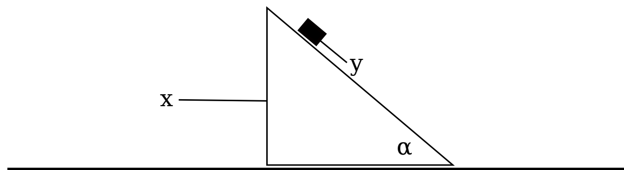
PHYS-3203 Homework 5 Due 15 Feb 2023

This homework is due to `https://uwcloud.uwinnipeg.ca/s/NwC99SeB7qHz9Ky` by 10:59PM on the due date. Your file(s) must be in PDF format; they may be black-and-white scans or photographs of hardcopies (all converted to PDF), PDF prepared by LaTeX, or PDF prepared with a word processor *using an equation editor*.

1. **Box on a Wedge Hamiltonian Version**

   Consider again the box of mass m sliding down a wedge of mass M on a frictionless horizontal surface. See the figure

   On the previous assignment, you should have found that the Lagrangian is

   $$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 - 2\dot{x}\dot{y}\cos\alpha\right) + mgy\sin\alpha \ . \tag{1}$$

   (a) Show that the velocities and canonical momenta are related by

   $$p_x = (M+m)\dot{x} - m\dot{y}\cos\alpha \ , \quad p_y = m(\dot{y} - \dot{x}\cos\alpha) \tag{2}$$

   and

   $$\dot{x} = \frac{p_x + p_y\cos\alpha}{M + m\sin^2\alpha} \ , \quad \dot{y} = \frac{p_y}{m} + \frac{p_x + p_y\cos\alpha}{M + m\sin^2\alpha}\cos\alpha \ . \tag{3}$$

   (b) Find the Hamiltonian. *Hint:* note that the term in parentheses in the Lagrangian can be written $p_y^2/m^2 + \dot{x}^2\sin^2\alpha$.

   (c) Name two conserved quantities in this system.

2. **Hamiltonian Central Force Motion** *expanded from Kibble & Berkshire*

   Consider an object of mass m moving in 3D with a central conservative force of potential energy $V(r)$. Note that we learned last semester that the kinetic energy in spherical coordinates is

   $$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) \ . \tag{4}$$

   (a) Write the Hamiltonian for this object in spherical polar coordinates.

   (b) You should see that the azimuthal angle ϕ is cyclic. Assuming motion is confined to the equatorial plane, find the effective potential for radial motion. Use this to argue that $p_\phi = L_z$, the z component of angular momentum. *Hint:* compare to the effective potential for motion in a central potential from PHYS-3202.

   (c) Define the square angular momentum

   $$\vec{L}^2 = m^2r^4\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) \ . \tag{5}$$

   Write $\vec{L}^2$ in terms of canonical momenta and show that it is conserved, even though θ is not cyclic. *Hint:* Look at the class notes for the time dependence of a general function of positions and canonical momenta.

3. **Average Height**

A particle of mass m is moving near the surface of the earth with gravitational potential energy $V = mgz$, where z is the vertical direction. Assume the particle has energy E.

(a) Use the virial theorem to find its average height (value of z) over time.

(b) If the particle starts at rest at its maximum height, write the height z as a function of time. Assuming the particle stops when it reaches the ground at $z = 0$, calculate the time average of z. Is it the same as the answer from the previous part?