PHYS-3203 Homework 2 Due 18 Jan 2023

This homework is due to `https://uwcloud.uwinnipeg.ca/s/NwC99SeB7qHz9Ky` by 10:59PM on the due date. Your file(s) must be in PDF format; they may be black-and-white scans or photographs of hardcopies (all converted to PDF), PDF prepared by LaTeX, or PDF prepared with a word processor *using an equation editor.*

1. **More on the Brachistochrone**

   Consider a particle moving along a brachistochrone path, written parametrically as

   $$x = a(\theta - \sin\theta) \ , \quad y = a(1 - \cos\theta) \ , \tag{1}$$

   under the influence of gravity (without friction). Note that y increases downward. Show that the time for the particle to move from $x = y = 0$ to the minimum of the curve at $x = \pi a, y = 2a$ can be written as

   $$T = \frac{1}{\sqrt{2g}} \int_0^\pi d\theta \sqrt{\frac{(dx/d\theta)^2 + (dy/d\theta)^2}{y}} \tag{2}$$

   and show that the time for the object to reach the bottom is $T = \pi\sqrt{a/g}$.

2. **Geodesic on a Cone** *a very different version of a Kibble & Berkshire problem*

   A *geodesic* is the minimal length curve on a surface between two points on that surface (or possibly in a curved space). For example, we showed that a straight line segment is a geodesic on a plane, and you may know that a great circle is a geodesic on a sphere. Here we will examine geodesics on a cone with its tip at the origin and its axis of symmetry along the z axis. The surface of the cone is at a polar angle α from the z axis.

   (a) Find the relationship between the cylindrical coordinates r and z on the surface of the cone (this is a holonomic constraint). Then show that the distance L along a curve from point (r_1, θ_1) to point (r_2, θ_2) on the cone can be written

   $$L = \int_{r_1}^{r_2} dr \sqrt{\csc^2\alpha + r^2(\theta')^2} \tag{3}$$

   where $\theta' = d\theta/dr$, where the curve is described by the function $\theta(r)$ that gives angular position as a function of radius from the z axis. *Hint:* recall from PHYS-3202 that the infinitesimal Pythagorean distance in cylindrical coordinates is $ds^2 = dr^2 + r^2 d\theta^2 + dz^2$.

   (b) Find the Euler-Lagrange equation for $\theta(r)$ that minimizes the length along the curve.

   (c) Re-organize the Euler-Lagrange equation to show that

   $$\sin\alpha\,\theta' = \frac{a}{r\sqrt{r^2 - a^2}} \ , \tag{4}$$

   where a is a constant. Then integrate both sides with respect to r to show that the geodesic is given by the equation $r = a\sec[\sin\alpha(\theta - \theta_0)]$, where θ_0 is an integration constant. *Hint:* make the substitution $r = a\sec(\phi)$ to do the integral on the right-hand side of (4).