The Action + Lagrangian Mechanics

– The Action + Hamilton's Principle

- The Lagrangian function
  + Let's think about some key quantities of (Newtonian mechanics) by taking
  $$\vec{p} = m\dot{\vec{x}}, \quad T = \tfrac{1}{2}m\dot{\vec{x}}^2, \quad \vec{F} = -\vec{\nabla}V(\vec{x}) \text{ (when conservative)}$$
  + Newton's 2nd law is of course $\dot{\vec{p}} = \vec{F}$.
  But we have another relationship we've previously ignored: $p_i = \partial T/\partial \dot{x}_i$.
  + We can rewrite the 2nd law as $\frac{d}{dt}(\partial T/\partial \dot{x}_i) = -\partial V/\partial x_i$
  + We can define the Lagrangian function
  $$L = T - V \qquad (*)$$
  so the eqn of motion is
  $$d/dt\,(\partial L/\partial \dot{x}_i) - \partial L/\partial x_i = 0$$
  which takes the form of Euler-Lagrange eqns.
  + Although we are for now defining L as $(*)$ for normal conservative forces, and the usual v^2 kinetic energy, physicists often take L to be the fundamental quantity + allows more general functions. We'll see some later.

- Hamilton's principle
  + The fact that Newton's 2nd law can be recast as an E-L eqn, suggest we define a functional, the action
  $$S = \int_{t_0}^{t_f} dt\, L(t, \vec{x}, \dot{\vec{x}})$$

+ Hamilton's Principle (of least action) states that the actual path of a particle moving from $\vec{x}_0$ to $\vec{x}_f$ from time t_0 to t_f is the path that minimizes the action functional

+ In some cases, the physical path may be another type of extremum, but typically it is a minimum

+ Here's the advantage of minimizing an action vs using Newton's laws: the action/Lagrangian are scalars. We can use generalized coordinates immediately without having to worry about unit vectors in those directions, what the acceleration looks like, etc (think about spherical coords)

+ To deal with constraints, we can introduce generalized coordinates q_i or Lagrange multipliers → add appropriate terms to L

- Some interpretation:
  + Newtonian + Lagrangian mechanics are equivalent even though they are formulated differently. (General proof later.) Lagrangian mechanics gives a global meaning to the differential (local) form of Newtonian mechanics

  + The E-L eqns are essentially the 2nd law in general coordinates: define

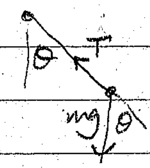
$$p_i = \partial L/\partial \dot{q}_i \equiv \text{canonical momentum for } q_i$$
(angular mom. if q_i = angle, etc)
$$Q_i = \partial L/\partial q_i \equiv \text{generalized forces}$$
(again, may be torques, etc)

then $\dot{p}_i = Q_i$

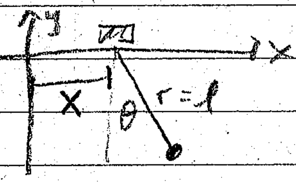
  + Lagrange multipliers typically contribute forces of constraint to Q_i → we'll see these later
  + Non-conservative forces like friction also add to Q_i but are not part of $\partial L/\partial q_i$

– Examples:

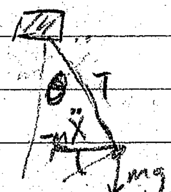
- Simple Pendulum: A bob a fixed radius $r = l$ from a support
  - The speed of the circular motion is $l\dot{\theta}$ in terms
  - of the polar angle
  - The potential is $V = mgl(1-\cos\theta)$, so Lagrangian is $L = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1-\cos\theta)$
  - The E-L equation is $ml^2\ddot{\theta} + mgl\sin\theta = 0$ as expected
  - We could also use a Lagrange multiplier to set $r = l$. Then the modified Lagrangian is
    $$L' = \tfrac{1}{2} m \dot{r}^2 + \tfrac{1}{2} m r^2 \dot{\theta}^2 - mg(l - r\cos\theta) - \lambda(r-l)$$
  - The EL eqn for θ is unchanged once we use $r = l = \text{const}$. from the λ eqn. The r E-L eqn is
    $mv^2/r = ml\dot{\theta}^2 = -mg\cos\theta + \lambda$.
    The Lagrange multiplier λ is equal to the tension keeping the bob on the circle!



- Pendulum on Moving support: previous example



  - The pendulum support has mass M and moves frictionlessly on a track along the x axis w/ position X
  - The bob is at fixed radius $r = l$ and angle θ from the vertical w.r.t. the instantaneous support position. Bob mass $= m$
    These are standard pendulum variables in the accelerating support frame.
  - In a Newtonian analysis, we need tension T of pendulum. We have



$$M\ddot{X} = T\sin\theta,$$
$$ml\ddot{\theta} = -mg\sin\theta - m\ddot{X}\cos\theta$$

centripetal acceleration

$$-ml\dot{\theta}^2 = -T - m\ddot{X}\sin\theta + mg\cos\theta$$

This includes the fictitious force in the ~~support's~~ support's frame.
We can eliminate T, then mg from 1st eqn

  - Alternately, we recall the bob's position is given by
    $x = X + l\sin\theta, \quad y = -l\cos\theta$
    $\Rightarrow \dot{x} = \dot{X} + l\cos\theta\,\dot{\theta}, \quad \dot{y} = +l\sin\theta\,\dot{\theta}$

So

$$L = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m\left(\dot{X}^2 + l^2\dot{\theta}^2 + 2l\dot{X}\dot{\theta}\cos\theta\right) + mgl\cos\theta$$

+ the eqn of motion are

$$\frac{d}{dt}\left[(M+m)\dot{X} + ml\dot{\theta}\cos\theta\right] = 0$$

$$\frac{d}{dt}\left[ml^2\dot{\theta} + ml\dot{X}\cos\theta\right] + ml\dot{X}\dot{\theta}\sin\theta + mgl\sin\theta = 0$$

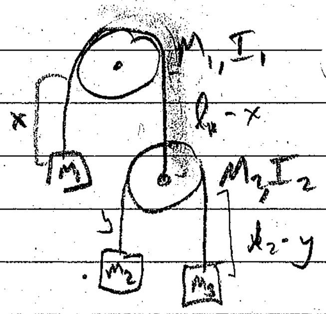
$$= ml^2\ddot{\theta} + ml\ddot{X}\cos\theta + mgl\sin\theta = 0.$$

This already has the simplifications above automatically — we never had to think about tension (force of constraint)

+ Can think about interpretation of EOM + check consistency with various limits

+ Note that the Lagrangian formalism with these coordinates automatically accounts for the accelerating frame. Also, b/c L does not depend on X, there is a conserved quantity

- Double Atwood Machine: This is a pulley connecting two masses plus another mass hanging from another pulley. What are the accelerations?



+ The constraints are that the strings over the pulleys are fixed length. So the positions of m_1 and pulley M_2 are x and $l_1 - x$, while the positions of the lower m_2 + m_3 masses are $l_1 - x + y$ and $l_1 - x + l_2 - y$. The angular position of pulley 1 is $\theta = x/R_1$, and the angular position of pulley 2 is $\phi = y/R_2$.

+ The kinetic energy is therefore

$$T = \tfrac{1}{2}m_1\dot{x}^2 + \tfrac{1}{2}M_2\dot{x}^2 + \tfrac{1}{2}m_2(\dot{y}-\dot{x})^2 + \tfrac{1}{2}m_3(\dot{y}+\dot{x})^2 + \tfrac{1}{2}I_1\dot{x}^2/R_1^2 + \tfrac{1}{2}I_2\dot{y}^2/R_2^2$$

and potential energy is

$$V = -m_1 g x - M_2 g(l_1 - x) - m_2 g(l_1 - x + y) - m_3 g(l_1 + l_2 - x - y)$$

so

$$L = \tfrac{1}{2}(m_1 + M_2 + I_1/R_1^2 + m_2 + m_3)\dot{x}^2 + \tfrac{1}{2}(m_2 + m_3 + I_2/R_2^2)\dot{y}^2 + (m_3 - m_2)\dot{y}\dot{x}$$
$$+ (m_1 - M_2 - m_2 - m_3)gx + (m_2 - m_3)gy + \text{const.}$$

+ The EOM are therefore

$$(m_1 + M_2 + m_2 + m_3 + \frac{I_1}{R_1^2})\ddot{x} + (m_3 - m_2)\ddot{y} = (m_1 - M_2 - m_2 - m_3)g$$
$$(m_2 + m_3 + I_2/R_2^2)\ddot{y} + (m_3 - m_2)\ddot{x} = (m_2 - m_3)g$$

We can solve for $\ddot{x}, \ddot{y}$ and then plug back into the 3 mass accelerations $\ddot{x}$, $\ddot{y} - \ddot{x}$, and $-\ddot{y} - \ddot{x}$.

- Spherical Pendulum: Like Foucault's pendulum, this is a pendulum allowed to swing in any direction at a fixed length l from a fixed support. Alternately, it is an object sliding w/o friction in a spherical bowl

+ If θ is the polar angle from the downward vertical axis, and ϕ is the azimuthal angle,

$$L = \tfrac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2) + mgl(1 - \cos\theta)$$

+ Since L does not depend on ϕ, the angular momentum
$$p_\phi = \partial L/\partial\dot{\phi} = m l^2 \sin^2\theta\, \dot{\phi}$$
is conserved.

+ We can now take 2 approaches to analyze the motion. One is to get the θ EL eqn

$$m l^2 \ddot{\theta} = m l^2 \dot{\phi}^2 \sin\theta\cos\theta - mgl\sin\theta$$
$$= -mgl\sin\theta + p_\phi^2 \cos\theta / m l^2 \sin^3\theta$$

You can then analyze in various approximations, such as a simple pendulum limit, oscillation around + precession of a circular orbit, etc.
You may notice that this follows from an effective potential if you integrate.

+ Be careful NOT to plug the angular momentum back into the Lagrangian first. Using

$$L = \tfrac{1}{2} m l^2 \dot{\theta}^2 + \left[\tfrac{1}{2} p_\phi^2 / m l^2 \sin^2\theta - mgl(1 - \cos\theta)\right]$$

gives the wrong EOM. Generally, only plug EOM back in special circumstances.

- Things to note:
  + With the right coordinates, we don't have to worry about things like normal forces, tensions, etc that only enforce constraints
  + If you use Lagrange multipliers, the value you find for them is the value of the constraint force.
  + If your coordinates correspond to use of an accelerating frame, the E-L eqns automatically include the fictitious forces
  + If L is independent of coordinate q, momentum $p = \frac{\partial L}{\partial \dot{q}}$ is conserved

– Equivalence with Newtonian Mechanics
- We already saw this for unconstrained Cartesian coords $\vec{x}$ and conservative forces. Let's see for general coords. For simplicity, we'll assume:
  + We can solve constraints by choice of coords (holonomic). Can generalize to Lagrange multipliers.
  + Assume that forces other than forces of constraint are conservative. Will discuss this at the end.
  + The system is natural, so $x_i(\{q\}, t)$ actually has $\partial x_i/\partial t = 0$. Can easily add this back

- Start with canonical momentum
  + With our assumptions, $L = T(q,\dot{q}) - V(q)$ where $T = \frac{1}{2} m \dot{\vec{x}}^2$ in Cartesian coordinates with $\vec{x} = \vec{x}(q)$
  + Therefore,
$$p_i = \partial L/\partial \dot{q}_i = \partial T/\partial \dot{q}_i = m \sum_j \dot{x}_j \, \partial \dot{x}_j/\partial \dot{q}_i$$
  + But $\dot{x}_j = \sum_i (\partial x_j/\partial q_i)\, \dot{q}_i \Rightarrow \partial \dot{x}_j/\partial \dot{q}_i = \partial x_j/\partial q_i$, so
$$p_i = m \sum_j \dot{x}_j \, \partial x_j/\partial q_i$$

- The E-L eqn then term
$$\frac{d}{dt}(\partial L/\partial \dot{q}_i) = dp_i/dt = m \sum_j \left(\ddot{x}_j \, \partial x_j/\partial q_i + \dot{x}_j \frac{d}{dt}\frac{\partial x_j}{\partial q_i}\right)$$
$$= m \sum_j \ddot{x}_j \, \partial x_j/\partial q_i + m \sum_{j,k} \dot{x}_j \frac{\partial^2 x_j}{\partial q_k \partial q_i} \dot{q}_k$$

+ The 1st of these is given by the force

$$m \sum_j \ddot{x}_j \, \partial x_j / \partial q_i = \sum_j F_j \, \partial x_j / \partial q_i$$

+ The 2nd term contains (by commutativity of partials)

$$\sum_k \frac{\partial^2 x_j}{\partial q_k \partial q_i} \dot{q}_k = \sum_k \frac{\partial^2 x_j}{\partial q_i \partial q_k} \dot{q}_k = \frac{\partial}{\partial q_i}\left(\sum_k \frac{\partial x_j}{\partial q_k} \dot{q}_k\right) = \frac{\partial \dot{x}_j}{\partial q_i}$$

Therefore, the 2nd term is $\sum_j m \dot{x}_j \, \partial \dot{x}_j / \partial q_i = \partial T / \partial q_i$

- What are the forces?
  + Divide $\vec{F}$ into conservative forces $-\vec{\nabla} V$ and constraint forces $\vec{F}'$
  + Constraint forces for holonomic natural constraints act to stop the coordinates from moving off surface $\vec{x}(q)$, so they act $\perp$ to the surface. But $\partial x_j / \partial q_i$ are always tangent to the surface
  + Therefore

$$\sum_j F_j \, \partial x_j / \partial q_i = \sum_j \left(F'_j - \partial V / \partial x_j\right) \partial x_j / \partial q_i$$

$$= -\partial V / \partial q_i = Q_i \text{ generalized force.}$$

- Altogether, we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = -\frac{\partial V}{\partial q_i} + \frac{\partial T}{\partial q_i} = \partial L / \partial q_i$$

  + If there are nonconservative, non constraint forces $\hat{F}_j$, like kinetic friction, these don't follow from a Lagrangian. We have to modify the E-L eqn to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \hat{Q}_i$$

  where $\hat{Q}_i = \sum_j \hat{F}_j \, \partial x_j / \partial q_i$ (in the

  + Some nonconservative forces can be described by letting $V = V(q, \dot{q})$. See next! (you could also possibly generalize T)

- Electromagnetism
  - The Lorentz force is conservative only b/c the magnetic force does no work. But we can define a potential anyway
  - We recall / learn that

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla}\times\vec{A}$$

where Φ = (electric) scalar potential, $\vec{A}$ = vector potential

  - Consider the potential $V(\vec{x},\dot{\vec{x}}) = q\,\Phi(\vec{x},t) - q\dot{\vec{x}}\cdot\vec{A}(\vec{x},t)$
    We want to write the force
    + The

$$F_i = q(\vec{E} + \dot{\vec{x}}\times\vec{B})_i = -\frac{\partial V}{\partial x_i} + \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}_i}\right)$$

    + We can examine $x_i = x$ in Cartesian coords

$$\frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right) = -q\frac{dA_x}{dt} = $$
$$= -q\left(\partial A_x/\partial t + \dot{x}\,\partial A_x/\partial x + \dot{y}\,\partial A_x/\partial y + \dot{z}\,\partial A_x/\partial z\right)$$

    + Meanwhile,

$$\partial V/\partial x = q\,\partial\Phi/\partial x - q\left(\dot{x}\,\partial A_x/\partial x + \dot{y}\,\partial A_y/\partial x + \dot{z}\,\partial A_z/\partial x\right)$$

    + So

$$-\frac{\partial V}{\partial x} + \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right) = -q\left(\partial\Phi/\partial x + \partial A_x/\partial t\right)$$
$$+ q\left[\dot{y}\left(\partial A_y/\partial x - \partial A_x/\partial y\right) + \dot{z}\left(\partial A_z/\partial x - \partial A_x/\partial z\right)\right]$$
$$= +q E_x + q\dot{y}B_z - q\dot{z}B_y$$

    + This is indeed the right component of the force we want.

  - Therefore, in Cartesian coords, $\frac{d}{dt}\left(\partial T/\partial \dot{x}_i\right) = -\frac{\partial V}{\partial x_i} + \frac{d}{dt}\left(\partial V/\partial \dot{x}_i\right)$

$$\Rightarrow \; d/dt\left(\partial L/\partial \dot{x}_i\right) - \partial L/\partial x_i = 0$$

The proof readily extends to generalized coordinates as above