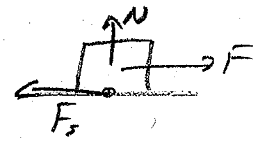
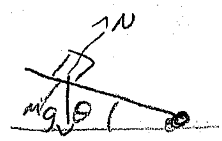
Linear Motion

- Gravity, Friction, Air Resistance in 1D
  - Review
    - Near the surface of the earth, gravity exerts an approx. constant force $\vec{g}$ down ($g \approx 9.8\ m/s^2$)
    - Friction is a contact force between objects proportional to the normal force between them. Always opposes relative motion

      + Static friction $F_s \leq \mu_s N$ when no relative motion. It always matches opposing force
      + Kinetic friction $F_k = \mu_k N$ against relative velocity
    - Air resistance (or more generally fluid resistance) is directed against velocity relative to the air but typically $F = \lambda v^n$ where v = relative speed
  - Example 1. (Motion in 1D, forces in 2D)

    A block is at rest on a liftable ramp. The coefficient of static friction is μ_s. How large an angle θ do we need to raise the ramp to cause the block to slide
    - First, work out the forces. Choose axes along the ramp and $\perp$.
      + Gravity is $mg \sin\theta$ down ramp, friction $-F_s$ (ie, up)
      + Normal force is N up right, gravity is $-mg\cos\theta$ (down left)
    - There is no acceleration (or motion) into the ramp, so $N = mg\cos\theta$
    - Along the ramp, $m\ddot{x} = mg\sin\theta - F_s = 0 \Rightarrow mg\sin\theta = F_s \leq \mu_s mg\cos\theta$

      $\Rightarrow \tan\theta \leq \mu_s$. Or, $\theta = \tan^{-1}\mu_s$ when motion starts

- Example 2

Consider the same block + same ramp with coefficient of kinetic friction μ_k. What is the acceleration?

- The forces down the ramp are $mg\sin\theta$ and $-\mu_k N = -\mu_k mg\cos\theta$
- So $m\ddot{x} = mg\sin\theta - \mu_k mg\cos\theta \Rightarrow \ddot{x} = g(\sin\theta - \mu_k\cos\theta)$

- Example 3: Terminal velocity

An object is falling directly downward opposed by air resistance.

- The total force down is $\vec{F} = m\vec{g} - \lambda|\vec{v}|^{n-1}\vec{v}$ where $\vec{v} = \dot{x}\hat{i}$ is + down.
- There is a maximal speed at which $F=0$ (terminal velocity). This is given by $v = (mg/\lambda)^{1/n}$
- For small enough velocities, $n=1$. Then

$$\dot{v} + \left(\frac{\lambda}{m}\right)v = g \Rightarrow v = \frac{mg}{\lambda}\left(1 - e^{-\lambda t/m}\right) \text{ for } v=0 \text{ @ } t=0.$$

x from integration

Collisions, etc

- Consider 2 objects of masses m_1 + m_2.
  - Reminder, Newton's 3rd Law is

$$m_1\ddot{x}_1 = -F_{21} \text{ and } m_2\ddot{x}_2 = F_{21} \Rightarrow \frac{d}{dt}(p_1+p_2)=0$$

Momentum is conserved (constant)

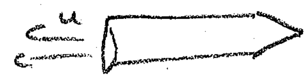
  - For initial velocities $u_{1,2}$ and final velocities $v_{1,2}$ $\Rightarrow$

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \Rightarrow m_1(u_1 - v_1) = m_2(v_2 - u_2)$$

  - Coefficient of restitution is ratio of relative speeds
    + $v_2 - v_1 \equiv e(u_1 - u_2)$ Signs are because initially they get closer ($u_2 - u_1 < 0$), then they get apart
    + Can solve for final velocities
    + $e=1$ is elastic. Typically $e<1$. $e>1$ only for something like an explosion

— Rocket Propulsion: use of momentum conservation

- Imagine a "collision" where initial relative velocity $= 0$, then 2 objects move apart (due to spring, explosion, etc).
  + A rocket is the limit where 1 object has vanishing mass
  + In other words, exhaust leaves rocket at speed u relative to rocket



- Suppose rocket has total mass m, velocity v
  + Loses mass $dm < 0$ to exhaust
  + Momentum conservation

$$mv = (m+dm)(v+dv) + \underbrace{(-dm)(v-u)}_{\text{exhaust}}$$

$$\Rightarrow \quad m\dot{v} + \dot{m}u = 0 \quad \text{absent ext. forces}$$

  + $-\dot{m}u$ plays the role of force. Called thrust. $\dot{m} < 0$, so thrust is positive.
  + Solution: $v = v_0 + u\ln(m_0/m)$. (m_0, v_0 = initial values)
- See reading about stages + gravity

⑥ Energy + Conservation

— Derivation

- Multiply 2nd law by $\dot{x}$, or $m\dot{x}\ddot{x} = F\dot{x}$
- This is $\frac{d}{dt}(T)$, where $T \equiv \frac{1}{2}m\dot{x}^2$ is kinetic energy
- Change in KE is $\int F dx'$ with F evaluated as it is on object's path of motion
  ↖ work done by F

— Conservation of energy

- If F depends only on position x, we can define potential energy $V(x) = -\int F(x)\,dx$ ← constant of integration sets zero.
- Then Newton's 2nd law says $E = T + V$ is conserved or constant.
  + Such forces are conservative
  + Friction, drag, etc are dissipative. Energy in these

cases is still conserved in total if we include heat (or eventually electromagnetic fields.)

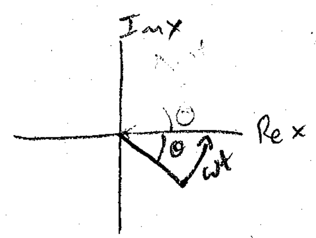
- Thinking back:
  - Elastic collisions conserve kinetic energy. Start with this.
    - Recall momentum $m_1(u_1 - v_1) = m_2(v_2 - u_2)$
    - KE conservation is $\frac{1}{2}m_1(u_1^2 - v_1^2) = \frac{1}{2}m_2(v_2^2 - u_2^2)$
    - Dividing, $u_1 + v_1 = u_2 + v_2$, or $u_1 - u_2 = v_2 - v_1 \Rightarrow e = 1$.
  - Gravity has potential energy $V(x) = mgx$, where x increases up.
    - A ball dropped from height h has velocity given by $\frac{1}{2}mv^2 = mgh$ at $x = 0$.
    - More generally, $|v| = \sqrt{\frac{2E}{m} - 2gx}$ for motion in gravity.
    - Can combine this with usual constant force solution $x = x_0 + v_0 t - \frac{1}{2}gt^2$ as appropriate

Harmonic Oscillators

- Importance
  - You may remember these have $F(x) = -kx$, $V(x) = \frac{1}{2}kx^2$. Why so important?
  - Consider a particle near equilibrium point (chosen to be $x=0$), which means $F(0) = \frac{-dV}{dx} = 0$.
    - For small motion, $V(x) \approx V(0) + V'(0)x + \frac{1}{2}V''(0)x^2 + \dots$
    - $V(0)$ does not contribute to force, $V'(0) = 0$ by assumption.
    - So $V(x) = \frac{1}{2}kx^2$ $(k = V''(0))$ is a good approximation near an equilibrium point for (almost) any potential.
  - For total energy E, motion is oscillation between $x = \pm\sqrt{2E/k}$. Max speed is $v = \pm\sqrt{2E/m}$ at $x = 0$, assuming $k > 0$ ($x = 0$ is a minimum).
  - Example: Charge q moving between 2 identical fixed charges at $x = \pm a$.
    $V(x) = \frac{q^2}{4\pi\epsilon_0}\left(\frac{1}{a-x} + \frac{1}{x+a}\right) \approx \dots$. Also pendulum

— Solution via ODE

- Newton's 2nd law gives $\ddot{x} + (k/m)x = 0$.
  - + 2nd order eqn, so should be 2 indep. sol'ns.
  - + Linear, so any linear superposition of sol'ns is a solution
- Guess a solution $x = Ae^{pt}$ for A, p constants.
  - + For $k<0$, the equilibrium is at a max. of $V(x)$.
    We find $p = \pm\sqrt{|k|/m}$, or $x = Ae^{pt} + Be^{-pt} \to Ae^{pt}$ as $t\uparrow$.
    Makes sense for unstable equilibrium.
  - + For potential min $k>0$, $p = \pm i\sqrt{k/m} \equiv \pm i\omega$.
  - + Solution $x = Ae^{i\omega t} + Be^{-i\omega t}$. Often just write $x = Ae^{i\omega t}$
- Meaning of complex solution.
  - + Of course, the true solution $x(t)$ must be real.
    By linearity, the real + imaginary parts of $Ae^{i\omega t}$ are solutions.
  - + For $A = \frac{1}{2}ae^{-i\theta}$, a real, these are $x = a\cos(\omega t - \theta)$
    and $x = a\sin(\omega t + \theta)$



  - + ω = (angular) frequency of motion
    θ = phase of motion.
    Period of motion $T = 2\pi/\omega$.

— Damped Oscillators

- Suppose an oscillator is also subject to air resistance. At low velocities, this is an additional force $F = -\lambda\dot{x}$
  (Note: friction is actually a bit more complicated)
- 2nd law gives
  $$\ddot{x} + \left(\frac{\lambda}{m}\right)\dot{x} + \left(\frac{k}{m}\right)x = 0$$
  - + This eqn also appears other places in physics, as in description of LRC circuits

+ Guess $x = Ae^{pt}$. Then $p^2 + \frac{\lambda}{m}p + \frac{k}{m} = 0 \Rightarrow p = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$

where $\gamma = \frac{\lambda}{2m}$, $\omega_0 = \sqrt{k/m}$ = natural or undamped frequency.

- 3 cases (given some initial conditions)
  + <u>Overdamped</u> $x = Ae^{-\gamma_+ t} + Be^{-\gamma_- t}$, $\gamma_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2}$ for $\gamma > \omega_0$

    Solution dies off exponentially. At late times, γ_- controls lifetime.
  + <u>Underdamped</u> If $\omega_0 > \gamma$, the power is complex $p = -\gamma \pm i\sqrt{\omega_0^2 - \gamma^2}$

    Solution is oscillation of frequency $\bar{\omega} = \sqrt{\omega_0^2 - \gamma^2}$ w/exp. envelope
    $x = Ae^{-\gamma t} e^{i\bar{\omega}t}$ (or real/imaginary part)
  + <u>Critically damped</u> $\omega_0 = \gamma$. The power is a double root, so the solution is $x = (A + Bt)e^{-\gamma t}$. The exponential decay is faster than γ_- of any overdamped case.

— Forced (Driven) Oscillators: Adding an additional external force

- Let's take an extra force $F(t) = Fe^{i\omega t}$ (really cos or sin)
  + The EOM is now $\ddot{x} + \frac{\lambda}{m}\dot{x} + \frac{k}{m}x = \frac{F}{m}e^{i\omega t}$
  + Take $x = Ae^{i\omega t}$ as a particular solution. Then
    $$A\left(-\omega^2 + 2i\gamma\omega + \omega_0^2\right) = F/m$$
  + Taken $A = |A|e^{-i\theta}$. In magnitude $|A| = \frac{F/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}$

    and in phase $\tan\theta = 2\gamma\omega / \omega_0^2 - \omega^2$. Assuming $F > 0$, $0 \le \theta \le \pi$
  + The general sol'n needed for initial conditions includes the two appropriate un-forced sol'ns from above.
- Physics of the solution:
  + The un-forced parts are <u>transients</u> that die off exponentially
  + θ is the <u>phase lag</u>: Once per period, $F(t)$ starts increasing. How much of a period later does x start increasing? $\theta = 0$ at $\omega = 0$ (in phase) and goes to π for $\omega \to \infty$ (out of phase).

+ Resonance: For a given oscillator, the frequency $\omega = \sqrt{\omega_0^2 - 2\gamma^2}$ that maximizes the amplitude is resonance. For underdamped $\omega \approx \omega_0$. Resonance gives a peak in amplitude with half-width γ. The quality factor $Q = \omega_0/2\gamma$ measures sharpness of peak and is ratio of amplitude at $\omega = \omega_0$ to $\omega = 0$. Important in circuits + engineering (examples).

- More general forcing:
  + Suppose $F(t)$ is periodic with period T. We can write it as a Fourier series

$$F(t) = \sum_{n=-\infty}^{\infty} F_n e^{in\omega t} \quad \text{with } \omega = \frac{2\pi}{T} \text{ and } F_n = \frac{1}{T}\int_0^T dt\, F(t) e^{-in\omega t}$$

By linearity of the EOM,

$$x(t) = (\text{transients}) + \sum_n \frac{F_n/m \; e^{in\omega t - i\theta_n}}{\sqrt{(\omega_0^2 - n^2\omega^2)^2 + 4\gamma^2 n^2\omega^2}}$$

Unless F_n gets larger for large $|n|$, the amplitude coefficients fall off.

Example Square wave force of period T with $F(t) = F$ for $0 \le t < T/2$ and $= 0$ for $T/2 \le t < T$. Then $F_n = (F/i\pi n)[1-(-1)^n]$

  + Similarly, we can solve for the response to a general $F(t)$ using a Fourier transform (review from Math. Phys.)

  + Alternately, consider the response to $F(t) = \Delta p\, \delta(t)$. The sol'n is the same as the transient for init. cond. $x = 0$, $\dot{x} = \Delta p/m$. Then build up any force as a superposition (integral) of δ-function forces. See KB.