PHYS-3202 Homework 2 Due 6 Oct 2021

This homework is due to `https://uwcloud.uwinnipeg.ca/s/wxqoYpEEa8WT2LX` by 10:59PM on the due date. Your file(s) must be in PDF format; they may be black-and-white scans or photographs of hardcopies (all converted to PDF), PDF prepared by LaTeX, or PDF prepared with a word processor *using an equation editor.*

1. **Exponential Forcing**

   Consider a damped harmonic oscillator with natural frequency ω_0 and damping γ. Suppose that it experiences a driving force $F(t) = F\exp(-\alpha t)$ (with F, α real) and satisfies the initial conditions $x(0) = 0, \dot{x}(0) = 0$.

   (a) Show that the function $x(t) = A\exp(-\alpha t)$ solves the equation of motion and find A. *Hint:* Newton's second law can be written as

   $$m\ddot{x} + 2m\gamma\dot{x} + m\omega_0^2 x = Fe^{-\alpha t} \ . \tag{1}$$

   (b) Assuming $\gamma = 0$, show that the solution at very long times can be written as

   $$x(t) = -\frac{(F/m\omega_0)}{\sqrt{\alpha^2+\omega_0^2}}\cos(\omega_0 t + \theta) \ , \text{ where } \tan\theta = \alpha/\omega_0 \ . \tag{2}$$

   Hint: Find the homogeneous solutions and particular solution (from the last part) and note which ones last to long times. Also look for how the angle θ can appear.

2. **A Couple of Vector Identities**

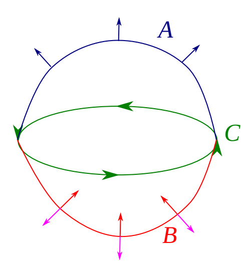
   (a) Using vector triple-product identities, write $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})$ in terms of the dot products $\vec{a}\cdot\vec{c}$, $\vec{b}\cdot\vec{c}$, $\vec{a}\cdot\vec{d}$, and $\vec{b}\cdot\vec{d}$

   (b) Verify the identity $\vec{\nabla}\times(\vec{a}\times\vec{b}) = (\vec{\nabla}\cdot\vec{b})\vec{a} + (\vec{b}\cdot\vec{\nabla})\vec{a} - (\vec{\nabla}\cdot\vec{a})\vec{b} - (\vec{a}\cdot\vec{\nabla})\vec{b}$ by comparing the z component of each side of the identity.

3. **Integral Theorems**

   In the figure, the line integral of a vector function $\vec{V}(\vec{x})$ around curve C in the direction of the green arrows is equal to the surface integral of the curl $\vec{\nabla}\times\vec{V}$ over either surface A or surface B with normal vector $\hat{n}$ for each surface in the direction of the blue and red arrows respectively:

   $$\oint_C d\vec{r}\cdot\vec{V} = \int_A d\vec{S}\cdot(\vec{\nabla}\times\vec{V}) = \int_B d\vec{S}\cdot(\vec{\nabla}\times\vec{V}) \tag{3}$$

   This is Stokes's theorem. This problem will show how the second equality is related to the volume between the two surfaces A and B.

   (a) Show that the surface integral over B with the normal vector $\hat{n}' = -\hat{n}$ is minus the surface integral over B in (3). Note that the pink arrows represent $\hat{n}'$ in the figure.

(b) The previous part tells us that the integral over A minus the integral over B as given in (3) can be written as a single surface integral over a closed surface with the normal vector pointed outward:

$$\int_A d\vec{S} \cdot (\vec{\nabla} \times \vec{V}) - \int_B d\vec{S} \cdot (\vec{\nabla} \times \vec{V}) = \oint d\vec{S} \cdot (\vec{\nabla} \times \vec{V}) \ . \tag{4}$$

Use Gauss's theorem to write the closed surface integral as a volume integral and show that that integral vanishes. This shows that the two surface integrals in (3) are the same. *Hint:* recall that $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{V}) = 0$ for any well-defined $\vec{V}(\vec{x})$.

4. **Spherical Polar Unit Vectors**

In this problem, you will find the components of the spherical polar unit vectors $\hat{r}, \hat{\theta}, \hat{\phi}$ in terms of the Cartesian unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$.

(a) We know that the velocity of an object is given by $\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k}$. Begin by finding expressions for $\dot{x}$, $\dot{y}$, and $\dot{z}$ in terms of spherical polar coordinates r, θ, ϕ and their time derivatives.

(b) By comparing to the form of $\vec{v}$ in spherical coordinates, find $\hat{r}, \hat{\theta}, \hat{\phi}$.