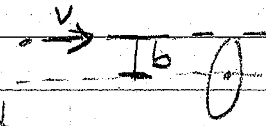
Scattering + Cross Sections

- Cross Sections + 3D Scattering / Collisions

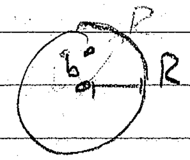
- How likely is it for 2 objects to hit each other in 3D? Consider the case where the target is very massive and doesn't move

  + For the case of force only on contact, this answer is the cross section area σ, or the area of the target's profile $\perp$ incoming object's velocity

  + We define the impact parameter b as the distance between the line parallel to the initial (far away) velocity and the target center

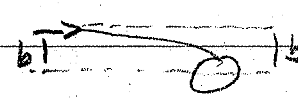


  + If the target is a sphere of radius R, they hit if $b \leq R$ and $\sigma = \pi R^2$. We will stick to the spherically symmetric case



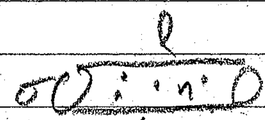
- With a nonzero central potential, the cross section is not geometric

  + In gravity, the cross section to hit earth is larger than R due to the attraction. $\sigma = \pi b_{max}^2$, where $b \leq b_{max} \Rightarrow$ collision



  + There is a scattering cross-section. Any time the incoming object changes direction is a scattering event.

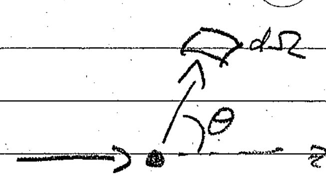
  + Suppose we attach the cross section to the object moving through number density n of targets. The mean free path l is the distance travelled w/ 1 collision on average $n\sigma l \equiv 1$



-Differential Cross Section

- How much does the incoming object scatter?
- Spec

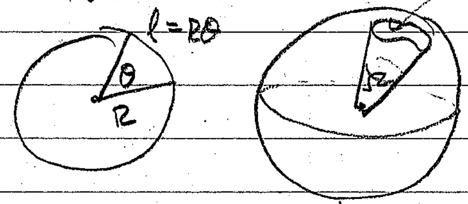
  + If initial velocity is on z axis w/target at origin, final velocity is given by spherical angles θ, ϕ. Scattering angle is θ. Think about



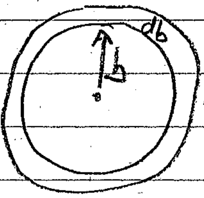
  + Suppose we catch all outgoing particles at θ, ϕ in a bucket of angular dimensions $d\theta, d\phi$ as seen from the origin.

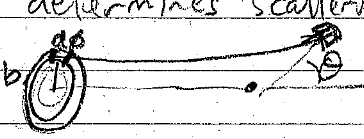
Think about the sky.

  + This bucket takes up a solid angle $d\Omega = \sin\theta \, d\theta \, d\phi$. Solid angle is defined as the area of a region on a sphere divided by radius squared. Similar to angle and arc length



  + The differential cross section $d\sigma/d\Omega$ is the area of impact parameter per solid angle. Note that impact parameter determines scattering angle





  + The total cross section is the integral over the sphere ("sky")

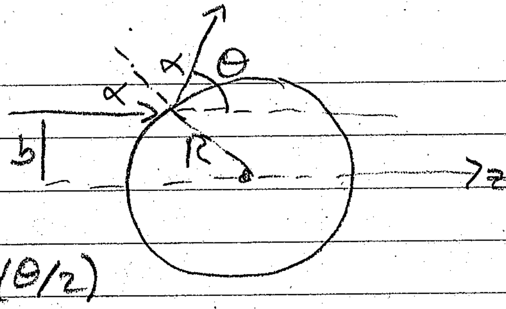
$$\sigma = \int d\Omega \, d\sigma/d\Omega$$

- Example Hard-Sphere Scattering

  + Target is a sphere of radius R. The total cross section is $\sigma = \pi R^2$ as above

  + For a hard sphere, the incoming particle scatters elastically. That means the particle "reflects" around the normal vector to the sphere's surface

+ We get the relation between $b + \theta$ geometrically. From the diagram, the reflection angle α is given by $\sin\alpha = b/R$, and $\theta = \pi - 2\alpha \Rightarrow b = R\cos(\theta/2)$



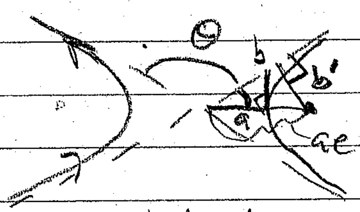
+ That means the cross sectional area is $d\sigma = |db|(b\,d\phi)$. From above, $db = -\frac{1}{2}R\sin(\theta/2)$. This is negative b/c larger impact parameters scatter less.

+ Altogether, $d\sigma = \frac{1}{4}R^2(\sin\theta\,d\theta\,d\phi) \Rightarrow d\sigma/d\Omega = R^2/4$. This is <u>isotropic</u> (independent of angle). Since the ~~total solid angle of a sphere is 4π~~

- <u>Example</u>: Rutherford Scattering

+ Scattering from an inverse square force. Rutherford's experiment scattered α particles (small particles of + charge) from gold nuclei (heavy particles of + charge). This is repulsive Coulomb scattering, but the cross section is the same for repulsive or attractive.

+ We can figure out the impact parameter and scattering angle from hyperbolic orbits.
Let b' = impact parameter;
$a + b$ = semi-major/minor axes
We know $a^2 + b^2 = a^2e^2$,



so the 2 triangles are congruent $\Rightarrow b' = b$.
(Note that this works w/ either branch of the hyperbola.)

+ The scattering angle comes from $\tan^{-1}(b/a) = \frac{1}{2}(\pi - \theta)$
Then
$$b = a\cot(\theta/2) = \frac{l}{e^2-1}\cot(\theta/2)$$
where
$$e^2 - 1 = \frac{2J^2E}{mk^2} = \frac{2lE}{|k|} \Rightarrow b = \frac{|k|}{mv^2}\cot(\theta/2)$$
where v = initial asymptotic speed.

+ The cross sectional area is $d\sigma = |db|(b\,d\phi)$ where
$db = -(|k|/2mv^2)\csc^2(\theta/2)\,d\theta$.
Then
$$d\sigma = \frac{k^2}{2(mv^2)^2}\frac{\cos(\theta/2)}{\sin^3(\theta/2)}\,d\theta\,d\phi.$$
With $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$,
$$\frac{d\sigma}{d\Omega} = \frac{1}{4}\left(\frac{k}{mv^2}\right)^2\frac{1}{\sin^4(\theta/2)}$$
This is the Rutherford scattering cross section (with $k = q_1 q_2/4\pi\epsilon_0$ for Coulomb potential)

+ The total cross section is infinite b/c the inverse square force is infinite range, so there is always a little scattering.

+ Rutherford's experiment found this cross section, which shows the nucleus is small (point like). If the nucleus were large, we would have to modify the force experienced at close distances so the cross section would be different at small b (large θ).