PHYS-4602 Homework 7 Due 17 Mar 2022

This homework is due to `https://uwcloud.uwinnipeg.ca/s/yPzo5AdxJx4oCMn` by 10:59PM on the due date. Your file(s) must be in PDF format; they may be black-and-white scans or photographs of hardcopies (all converted to PDF), PDF prepared by LaTeX, or PDF prepared with a word processor *using an equation editor.*

1. **Relativistic Harmonic Oscillator**

   Recall that the relativistic energy is $\sqrt{(\vec{p}c)^2 + (mc^2)^2} \approx mc^2 + \vec{p}^2/2m - \vec{p}^4/8m^3c^2$ (plus higher-order corrections), so a 1D harmonic oscillator with the first relativistic correction has the Hamiltonian

   $$H = \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \frac{1}{2}m\omega^2 x^2 \ . \tag{1}$$

   (a) Find the ground state energy of this oscillator using first-order perturbation theory. What condition must the frequency satisfy for the relativistic correction to be small?

   (b) Find the ground state eigenstate of this oscillator in terms of the unperturbed oscillator eigenstates using first-order perturbation theory.

2. **Stark Effect** *based on G&S 7.45*

   The presence of an external electric field $E_0\hat{z}$ shifts the energy levels of a hydrogen atom, which is called the Stark effect. Consider the hydrogen atom to be described by the Coulomb potential; the external electric field introduces a perturbation

   $$H_1 = eE_0 z = eE_0 r\cos\theta \ . \tag{2}$$

   We have already seen on homework that the expectation value of this Hamiltonian in the ground state $n = 1$ vanishes, so there is no shift in the ground state energy. In this problem, we consider the degenerate perturbation theory of the $n = 2$ states. *As spin does not enter, do not consider it in this problem.*

   (a) The four states $|2,0,0\rangle$, $|2,1,0\rangle$, and $|2,1,\pm1\rangle$ are degenerate at 0th order. Label these states sequentially as $i = 1,2,3,4$. Show that the matrix elements $W_{ij} = \langle i|H_1|j\rangle$ form the matrix

   $$W = -3aeE_0 \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \ , \tag{3}$$

   where empty elements are zero and a is the Bohr radius. *Hint*: Note that L_z commutes with H_1, so only states with the same quantum number m can have nonzero matrix elements; this will save you quite a bit of work. Then use the angular wavefunctions to see that all the diagonal elements of W must vanish. Finally, use the explicit wavefunctions to evaluate the remaining matrix elements of W (there should only be one independent one left).

   (b) Diagonalize this matrix to show that $|\pm\rangle = (1/\sqrt{2})(|2,0,0\rangle \pm |2,1,0\rangle)$ are eigenstates of W. Find the first order shift in energies of $|\pm\rangle$. *Hint*: Note that the corrected eigenstates may still have contributions from other values of the principal quantum numbers n, but that doesn't quite matter.

(c) Finally, show that the states $|\pm\rangle$ have a nonzero dipole moment $p_z = -e\langle z\rangle$ and calculate it. You should not need to do any more calculations; just use your answer from part (b).

3. **Matrix Perturbation Theory**

Consider the matrix Hamiltonian

$$H \simeq \begin{bmatrix} E_1 & \epsilon \\ \epsilon & E_2 \end{bmatrix} \tag{4}$$

with $E_1 \neq E_2$ except when you are told otherwise. Assume that $\epsilon \ll E_1, E_2$.

(a) To first order in perturbation theory, find the energy eigenvalues and eigenstates.

(b) What is the first order correction to the energy if $E_1 = E_2 = E$?

(c) Find the energy eigenvalues to second order in perturbation theory.

(d) Find the energy eigenvalues and eigenstates exactly. Then expand them as a power series in ϵ and compare to your perturbative answers from parts (a,c). In the case that $E_1 = E_2 = E$, how does your answer compare to part (b)?