PHYS-4602 Homework 4 Due 10 Feb 2022

This homework is due to `https://uwcloud.uwinnipeg.ca/s/yPzo5AdxJx4oCMn` by 10:59PM on the due date. Your file(s) must be in PDF format; they may be black-and-white scans or photographs of hardcopies (all converted to PDF), PDF prepared by LaTeX, or PDF prepared with a word processor *using an equation editor*.

1. **An Entropy Proof**

   Use concavity of the von Neumann entropy to prove that the von Neumann entropy of the maximally mixed density matrix $\rho_{MM} = \mathbb{I}/k$ is greater than that of any other density matrix ρ, where k is the dimension of Hilbert space and $\mathbb{I}$ is the $k \times k$ identity matrix. *Hint:* The von Neumann entropy depends only on the eigenvalues of the density matrix and is independent of basis. Use this fact to write multiple density matrices with the same entropy as ρ before applying concavity.

2. **Partial Traces and Entropy**

   Consider a system of two qubits with density operator

   $$\rho = \frac{1}{2}(|0\rangle_1|0\rangle_2)({}_1\langle 0|_2\langle 0|) + \frac{1}{2}(|+\rangle_1|1\rangle_2)({}_1\langle +|_2\langle 1|) \ , \tag{1}$$

   where $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$.

   (a) Find the reduced density operator and von Neumann entropy of qubit #2.

   (b) Find the reduced density operator and von Neumann entropy of qubit #1. Is the entropy the same as in part (a)? Why or why not?

   (c) Suppose the system actually has a third qubit such that the state of the 3-qubit system is a pure state (that is, the 3-qubit system is a purification of ρ). Write some example of such a 3-qubit pure state. What is the entropy of the third qubit considered on its own?

3. **2-Qubit Gates**

   Consider a 2 qubit system. Choose a basis for the 2 qubit Hilbert space and use it for all parts of this problem.

   (a) Write the CNOT gate operator as a matrix in that basis and show that it is unitary.

   (b) Consider the 1 qubit gate NOT acting only on the first qubit of our two. Write this gate (call it NOT_1) as a matrix in your 2-qubit basis.

4. **Cloning Means FTL Communication** *based on a problem by Wilde*

   Suppose that Alice and Bob are at two ends of an EPR/Bell experiment. In other words, they are at rest with respect to each other and separated by 5 lightyears, and each receives one of a pair of entangled electrons with total spin state $s = 0$ simultaneously (in their common rest frame). By prior agreement, Alice measures either the S_z or S_x spin of her electron as soon as she receives it, but Bob does not know which spin she measures.

   After Alice's measurement (in their rest frame time), Bob's electron is in some state $|\psi\rangle_B$. Suppose, in contradiction to the no-cloning theorem, Bob can clone his electron's state onto a large number N of other electrons. (For example, Bob can do some quantum operation that takes his $N+1$ electrons from state $|\psi\rangle_B|\uparrow\rangle_1 \cdots |\uparrow\rangle_N$ to state $|\psi\rangle_B|\psi\rangle_1 \cdots |\psi\rangle_N$.) What measurement(s) can Bob do on his extra N electrons that will tell him with great certainty whether Alice measured the S_z or S_x spin of her electron? Explain your answer. (Note that Bob can accomplish his measurement before Alice can tell him her measurement choice, so they can establish faster-than-light communication in this way. This is a good reason for the no-cloning theorem!)