PHYS-4602 Homework 3 Due 3 Feb 2022

This homework is due to `https://uwcloud.uwinnipeg.ca/s/yPzo5AdxJx4oCMn` by 10:59PM on the due date. Your file(s) must be in PDF format; they may be black-and-white scans or photographs of hardcopies (all converted to PDF), PDF prepared by LaTeX, or PDF prepared with a word processor *using an equation editor.*

1. **1-Qbit Density Matrix** *Griffiths & Schroeter 12.6 plus*

   Consider the density matrix ρ for a single qubit (you may consider this to be the spin of a single spin-1/2 particle instead). Here you will prove some properties that generalize to other systems.

   (a) Prove that $\rho^2 = \rho$ if and only if the state is pure. *Hint:* Think about the diagonal form of ρ as a matrix in pure and mixed states.

   (b) Suppose someone hands you a qubit and tells you it is 50% likely to be in either of the states $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. Write the density operator for this qubit as a dyad. Show that it is the same as if the other person told you the qubit was 50% likely to be in either state $|0\rangle$ or $|1\rangle$.

2. **Traces of Operators**

   Use Dirac bra/ket notation and the completeness relation for an orthonormal basis to prove the following (you may assume that the Hilbert space is finite-dimensional):

   (a) $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for any two operators A, B

   (b) The expectation value $\langle A\rangle = \mathrm{Tr}(A\rho)$ for any observable A in a mixed state with density operator ρ

3. **Total Spin and Mixed States**

   Consider a system composed of two spins of $s = 1$ combined into a total spin 2 state

$$|s=2, m=0\rangle = \frac{1}{\sqrt{6}}|1,1\rangle_1|1,-1\rangle_2 + \sqrt{\frac{2}{3}}|1,0\rangle_1|1,0\rangle_2 + \frac{1}{\sqrt{6}}|1,-1\rangle_1|1,1\rangle_2 \ . \tag{1}$$

   Write the reduced density operator for the first spin after the partial trace of the second spin. Is this a mixed state? Is it maximally mixed?

4. **Thermal State**

   The density matrix for a quantum system in thermal equilibrium at temperature T is

$$\rho = e^{-H/k_BT}/Z \text{ where } Z \equiv \mathrm{Tr}\left(e^{-H/k_BT}\right) \tag{2}$$

   is the *partition function*, H is the Hamiltonian, and k_B is the Boltzmann constant.

   (a) First, show that the partition function is

$$Z = \sum_n e^{-E_n/k_BT} \ , \tag{3}$$

   where the E_n are the energy eigenvalues.

(b) Show that another expression for the density operator is

$$\rho = \frac{1}{Z} \sum_n e^{-E_n/k_B T} |E_n\rangle\langle E_n| \ , \tag{4}$$

where $|E_n\rangle$ are the energy eigenvectors.

(c) Show that the so-called *thermofield double state*

$$|\psi\rangle = \frac{1}{\sqrt{Z}} \sum_n e^{-E_n/2k_B T} |E_n\rangle_1 |E_n\rangle_2 \ , \tag{5}$$

where $1, 2$ indicate two copies of the quantum system, is a purification of the thermal state by showing that the partial trace of $|\psi\rangle\langle\psi|$ over system 2 is ρ.