PHYS-4602 Homework 1 Due 20 Jan 2022

This homework is due to `https://uwcloud.uwinnipeg.ca/s/yPzo5AdxJx4oCMn` by 10:59PM on the due date. Your file(s) must be in PDF format; they may be black-and-white scans or photographs of hardcopies (all converted to PDF), PDF prepared by LaTeX, or PDF prepared with a word processor *using an equation editor.*

1. **The Last Eigenvector**

   A system with a three-dimensional Hilbert space has an operator represented by the matrix

$$B \simeq \frac{b}{3}\begin{bmatrix} 1/2 & -1/2 & 2i \\ -1/2 & 1/2 & -2i \\ -2i & 2i & -1 \end{bmatrix} \tag{1}$$

   in some basis. Two of the eigenstates are represented by the column vectors

$$|1\rangle \simeq \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } |2\rangle \simeq \frac{1}{\sqrt{3}}\begin{bmatrix} i \\ -i \\ 1 \end{bmatrix} . \tag{2}$$

   (a) Find the third eigenstate $|3\rangle$ as a column vector, including proper normalization. Use *only* relations between the eigenvectors, not the action of B on them. *Hint:* show that B is Hermitian first.

   (b) Find all the eigenvalues of B and write B as a matrix in the $\{|1\rangle, |2\rangle, |3\rangle\}$ basis.

   (c) The operator $|3\rangle\langle 3|$ projects any vector $|\psi\rangle$ onto its component in the $|3\rangle$ direction. Write $|3\rangle\langle 3|$ as a matrix in the original basis.

2. **Diagonalization** *Based on Griffiths A.26*

   Consider a three-dimensional Hilbert space with orthonormal basis $|e_i\rangle$, $i = 1, 2, 3$. The operator A takes the matrix representation

$$A = \sum_{i,j} |e_i\rangle\langle e_i|A|e_j\rangle\langle e_j| \simeq \begin{bmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{bmatrix} . \tag{3}$$

   You should be able to check yourself that A is Hermitian.

   (a) Find the eigenvalues a_i and corresponding eigenstates $|a_i\rangle$ ($A|a_i\rangle = a_i|a_i\rangle$) written in terms of their components $\langle e_j|a_i\rangle$. Choose the eigenstates to form an orthonormal eigenbasis; that is, choose any ambiguities such that $\langle a_i|a_j\rangle = \delta_{ij}$.

   (b) As stated in class, A can be written in the form

$$A = \sum_i a_i |a_i\rangle\langle a_i| , \tag{4}$$

   where a_i are the eigenvalues and $|a_i\rangle$ are the eigenvectors of A. Verify that formula (4) gives the same operator as (3) when you plug in your answer to part (a) for the eigenvalues and eigenvectors.

   (c) Write the state $|\psi\rangle = |e_1\rangle - i|e_3\rangle$ in the A eigenbasis (as a superposition of the $|a_i\rangle$).

3. **Commutators and Functions of Operators**

(a) Suppose $|a\rangle$ is an eigenfunction of some operator A, $A|a\rangle = a|a\rangle$. Consider the inverse operator A^{-1} defined such that $AA^{-1} = A^{-1}A = 1$. Show that $|a\rangle$ is an eigenvector of A^{-1} with eigenvalue $1/a$ if $a \neq 0$ (if there is an eigenvalue $= 0$, A is not invertible).

(b) For any function $f(x)$ that can be written as a power series

$$f(x) = \sum_n f_n x^n \ , \tag{5}$$

we can define

$$f(A) = \sum_n f_n A^n \ , \tag{6}$$

where A^n denotes operating with A n times. Show that

$$f(A)|a\rangle = f(a)|a\rangle \ . \tag{7}$$

Does this result hold if the power series includes negative powers?

(c) For any three operators A, B, C, show that

$$[A, BC] = [A, B]C + B[A, C] \ . \tag{8}$$

(d) Then prove by induction that

$$[A, B^n] = n[A, B]B^{n-1} \ , \tag{9}$$

if $[A, B]$ commutes with B (for $n > 0$).