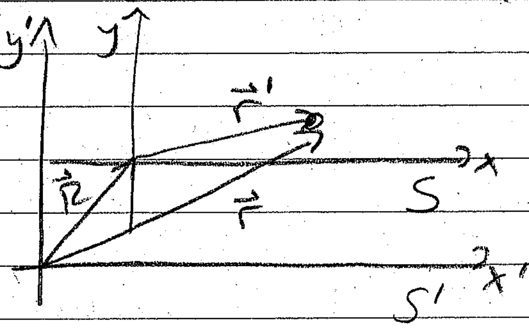
Systems of Particles + Center of Mass

General Principles

- Relativity Principle
  - Physics is the same in all inertial reference frames. That is, all forces in Newton's law (in an inertial frame) have a physical origin.
  - We can choose a convenient reference frame for solving a given physics problem.
  - We want to understand the relationship between a lab frame + the rest frame of the center of mass (CM frame) for an object or group of particles.

- Procedure for change of one reference frame to another
  - Work out change in positions.



+ If origin of frame S is at $\vec{R}$ according to frame S', the position of particle measured in the S' frame is related by

$$\vec{r}\,' = \vec{r} + \vec{R}$$

+ If S moves inertially w.r.t. S', $\vec{R} = \vec{R}_0 + \vec{V}t$.

  - Then you differentiate to find the relationship of velocities, combine to find KE, angular momentum, etc.

Center of Mass (CM) frame: frame where CM is at rest at the origin

- CM and momentum
  - Consider a group of particles of masses m_i and positions $\vec{r}_i$ in some lab frame S

+ The total mass is $M = \sum_i m_i$

+ The center of mass position is the mass-weighted average position

$$\vec{R} = \left(\sum_i m_i \vec{r}_i\right)/M$$

+ $\vec{R}$ is the position of the origin of CM frame S^* relative to lab frame S (ie, $\vec{R}^* = 0$)

+ For continuous materials, convert the sum to integrals

• Total Momentum

+ In lab frame, total momentum is

$$\vec{P} = \sum_i \vec{p}_i = \sum_i m_i \dot{\vec{r}}_i = \frac{d}{dt}\left(\sum_i m_i \vec{r}_i\right) = M\dot{\vec{R}}$$

This ~~is~~ momentum of one mass M particle moving with CM position $\vec{R}(t)$. Call this "CM particle"

+ Reminder

$$M\ddot{\vec{R}} = \dot{\vec{P}} = \sum \dot{\vec{p}}_i = \sum_i \left(\vec{F}_{ext,i} + \sum_j \vec{F}_{i,j}\right) = \vec{F}_{ext}$$

where $\vec{F}_{i,j}$ = force on particle i from particle j

and the double sum vanishes b/c $\vec{F}_{j,i} = -\vec{F}_{i,j}$ (3rd law)

+ In other words, total momentum is conserved if there are no external forces + the CM motion is determined by the external forces.

+ If the external force is uniform gravity,

$$\vec{F}_{i,G} = m_i \vec{g} \Rightarrow \vec{F}_G = M\vec{g} = M\ddot{\vec{R}}$$

$\Rightarrow$ Potential energy of uniform gravity is $V_G = -M\vec{g}\cdot\vec{R}$

– Energy of a group of particles

• Kinetic energy splits into CM contribution + CM frame

$$T = \frac{1}{2}\sum_i m_i \dot{\vec{r}}_i^{\,2} = \frac{1}{2}\sum_i m_i \left(\dot{\vec{r}}_i^{\,*} + \dot{\vec{R}}\right)^2$$

where

$\vec{r}_i^{\,*}$ = position in CM frame

+ Simplify
$$T = \frac{1}{2}\sum_i m_i\left(\dot{\vec{r}}_i^{*2} + \dot{\vec{R}}^2 + 2\dot{\vec{r}}_i^{*}\cdot\dot{\vec{R}}\right)$$

$$= T^* + \frac{1}{2}M\dot{\vec{R}}^2 + \left(\sum_i m_i\dot{\vec{r}}_i^{*}\right)\cdot\dot{\vec{R}}$$

where T^* = KE measured in CM frame.

+ The last term $= 0$ b/c $\sum m_i \vec{r}_i^* = \vec{R}^* =$ constant $(=0)$

+ Together, KE is "CM particle" energy plus energy in CM frame

- 2-particle special case $T = \frac{1}{2}M\dot{\vec{R}}^2 + T^*$
  + CM frame ~~KE~~ is
$$T^* = \frac{1}{2}m_1\dot{\vec{r}}_1^{*2} + \frac{1}{2}m_2\dot{\vec{r}}_2^{*2}$$
  + Define relative position
$$\vec{r} = \vec{r}_1^* - \vec{r}_2^* \quad \text{It} = \vec{r}_1 - \vec{r}_2 \text{ in lab frame too}$$
  + But for CM frame
$$m_1\vec{r}_1^* + m_2\vec{r}_2^* = 0 \Rightarrow \vec{r}_1^* = \frac{m_2\vec{r}}{M},\ \vec{r}_2^* = -\frac{m_1\vec{r}}{M}$$
  + Plug into KE + simplify
$$T^* = \frac{1}{2}m_1\left(\frac{m_2}{M}\right)^2\dot{\vec{r}}^2 + \frac{1}{2}m_2\left(\frac{m_1}{M}\right)^2\dot{\vec{r}}^2$$

$$= \frac{1}{2}\frac{m_1 m_2}{M}\dot{\vec{r}}^2 \equiv \frac{1}{2}\mu\dot{\vec{r}}^2$$

where $\mu \equiv m_1 m_2/M$ = reduced mass

  + Motion in CM frame is like motion of a single particle b/c the second one "mirrors" the first.

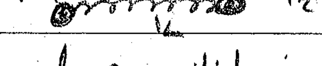
- Potential Energy
  + Forces split into external forces (from outside the system) and forces between particles in the system
$$\text{Force on particle } i = \vec{F}_{i,ext} + \sum \vec{F}_{ij}$$
  + If the internal forces are conservative, then

they come from a potential. By 3rd law + relativity principle, it must be central $V_{ij} = V_{ij}(|\vec{r}_i - \vec{r}_j|)$
The total is an internal potential energy $V_{int} = \sum V_{ij}$

+ ~~F~~

+ For rigid bodies, distances between particles don't change, so $V_{int} = \text{constant}$

+ We can work out $\frac{d}{dt}\left(T^* + V_{int}\right) = \sum_i \dot{\vec{r}}_i^* \cdot \vec{F}_{i,ext}$
Forces $\propto m_i$ like uniform gravity or fictitious forces cancel

• Example: 2 masses connected by spring 

+ The spring has spring constant k and equilibrium length d. So $V_{int} = \frac{1}{2}k(r-d)^2$

+ Suppose these are moving in 2D, so there is also gravity. Then

$$L = T - V = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - \frac{1}{2}k(r-d)^2 + M\vec{g}\cdot\vec{R}$$

$\Rightarrow$

$$M\ddot{\vec{R}} = +M\vec{g}, \quad \mu\ddot{\vec{r}} = -k(r-d)$$

+ Notice how the equations separate.

- Angular Momentum

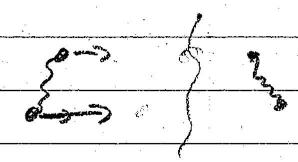
• Separation of CM contribution + CM frame value
+ We can re-write angular momentum as

$$\vec{J} = \sum_i m_i(\vec{r}_i^* + \vec{R}) \times (\dot{\vec{r}}_i^* + \dot{\vec{R}}) = \ldots$$
$$= M\vec{R}\times\dot{\vec{R}} + \sum m_i \vec{r}_i^* \times \dot{\vec{r}}_i^* \equiv M\vec{R}\times\dot{\vec{R}} + \vec{J}^*$$

+ This is angular momentum of CM and the angular momentum around the CM ($\vec{J}^*$)
+ For 2 particles, it again looks like one particle

$$\vec{J}^* = m_1\vec{r}_1^* \times \dot{\vec{r}}_1^* + m_2\vec{r}_2^* \times \dot{\vec{r}}_2^* = \mu\vec{r}\times\dot{\vec{r}}$$

- Example: 2 masses on spring again moving on a frictionless surface:
  - + If they slide differently, they rotate around each other in CM frame



  - + Total lab angular momentum is

$$\vec{J} = M\vec{R}\times\dot{\vec{R}} + \mu\vec{r}\times\dot{\vec{r}}$$

The 2nd term is how much they rotate around each other

  - + In cylindrical coords (ie, table in plane polar) for $\vec{r}$,

$$\vec{J}^* = \mu\vec{r}\times\dot{\vec{r}} = \mu\rho^2\dot{\varphi}\,\hat{z}$$

and

$$T^* = \tfrac{1}{2}\mu\dot{\vec{r}}^2 = \tfrac{1}{2}\mu\dot{\rho}^2 + \tfrac{1}{2}\mu\rho^2\dot{\varphi}^2 = \tfrac{1}{2}\mu\dot{\rho}^2 + \frac{J^{*2}}{2\mu\rho^2}$$

like we saw last term.

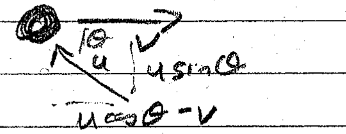
Applications + Examples

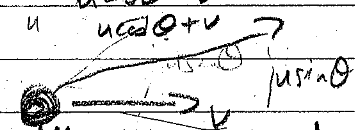
- Two-Body Collisions

- With no external forces, total momentum $M\dot{\vec{R}}$ is conserved.
  - + Kinetic energy is $T = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2$
  - + Since CM motion does not change, the max KE lost in an inelastic collision is $T^* = \frac{1}{2}\mu\dot{\vec{r}}^2$
  - + That's due to momentum conservation

- Elastic collision: KE conserved.
  - + Again, $T = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2$ with $\frac{1}{2}M\dot{\vec{R}}^2$ conserved
    $\Rightarrow \frac{1}{2}\mu\dot{\vec{r}}^2$ is also conserved
  - + This means the relative speed (magnitude of relative velocity $\dot{\vec{r}}$) is the same before & after the collision
  - + Example: In 1D, 2 equal mass objects collide with velocities $v_1 = 1\,m/s$, $v_2 = -2\,m/s$. They separate with velocities $u_1 = -2\,m/s$, $u_2 = 1\,m/s$ (Newton's Cradle)

- **Gravity Assist**: Space probe approaches a planet with relative speed u. Must leave w/same relative speed.
  - Planet is effectively infinitely massive, so CM frame is planet's rest frame.
  - In solar frame, planet has constant speed v "horizontally." Probe initial velocity components are $(u\cos\theta - v,\ u\sin\theta)$. To maintain same relative speed, final probe velocity is $(u\cos\theta + v,\ u\sin\theta)$.



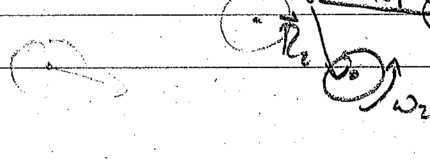


  - Useful for sending ships far away b/c of "free" speed boost
  - No need to do detailed orbit calculation

— Building Up a System

- We can group particles & build up KE & angular momentum in that way.
  - Suppose we are in total CM frame of 2 objects, which we take as being made of many particles. KE is $T = T_1 + T_2$, Ang. Mom is $\vec{J} = \vec{J}_1 + \vec{J}_2$
  - But if the total mass & CM position of object 1 are $M_1, \vec{R}_1$ we see that
    $$T_1 = \tfrac{1}{2}M_1\dot{\vec{R}}_1^{\,2} + T_1^*,\quad \vec{J}_1 = \tfrac{1}{2}M_1\vec{R}_1\times\dot{\vec{R}}_1 + \vec{J}^*$$
    like before. Same for object 2.
  - So you can think of groups of particles as an object and subdivide as much as you want

- **Example**: Two spinning hockey pucks
  - 1st puck CM is located at $\vec{R}_1$, and it is made of lots of individual atoms
  - Then
    $$T_1 = \tfrac{1}{2}\sum m\dot{r}^2 = \tfrac{1}{2}M_1\dot{\vec{R}}_1^{\,2} + T_1^* = \tfrac{1}{2}M_1\dot{\vec{R}}_1^{\,2} + \tfrac{1}{2}I_1\omega_1^2$$
    and
    $$\vec{J}_1 = M_1\vec{R}_1\times\dot{\vec{R}}_1 + I_1\omega_1\hat{z}$$
    (from what we know about rigid objects)



+ Repeat for 2nd hockey puck + combine

$$T = T_1 + T_2 = \left(\tfrac{1}{2}M_1\dot{\vec{R}}_1^2 + \tfrac{1}{2}M_2\dot{\vec{R}}_2^2\right) + \tfrac{1}{2}I_1\omega_1^2 + \tfrac{1}{2}I_2\omega_2^2$$

$$= \tfrac{1}{2}M\dot{\vec{R}}^2 + \tfrac{1}{2}\mu\dot{\vec{r}}^2 + \tfrac{1}{2}I_1\omega_1^2 + \tfrac{1}{2}I_2\omega_2^2$$

where M = total mass, $\vec{R}$ = total CM position, $\vec{r} = \vec{R}_1 - \vec{R}_2$

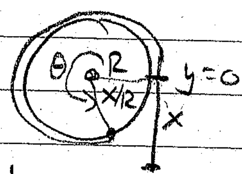
+ Also

$$\vec{J} = \vec{J}_1 + \vec{J}_2 = M\vec{R}\times\dot{\vec{R}} + \mu\vec{r}\times\dot{\vec{r}} + I_1\omega_1\hat{z} + I_2\omega_2\hat{z}$$

• Similar logic can work for non-rigid objects

Example: Unwinding Rope

+ A rope of linear density μ wraps once around a disk of radius R. Disk rotates around center to unwind the rope



+ When disk is at angle $\theta = 2\pi$, it's all wound up. Amount of hanging rope is $x = (2\pi - \theta)R$

+ Each piece of rope moves at speed $\dot{x}$ and disk rotates with $\dot{\theta} = -\dot{x}/R$. Kinetic energy is

$$T = \tfrac{1}{2}\mu l\dot{x}^2 + \tfrac{1}{2}I\dot{\theta}^2 = \pi R\mu\dot{x}^2 + \tfrac{1}{2}I\dot{x}^2/R^2$$

where $l = 2\pi R$ = length of rope

+ If we set center of disk at $y=0$, potential energy is given by CM of rope. This is located at

$$Y = \frac{1}{2\pi R\mu}\int dl\,\mu\,y = \frac{1}{2\pi R}\left[x\left(\frac{-x}{2}\right) + \int_0^{2\pi - x/R} d\theta R\,(R\sin\theta)\right]$$

given by CM of hanging part averaged w/ CM of wrapped part

Then

$$V = (2\pi R\mu)Yg = \mu g R^2\left[1 - \cos(x/R) - x^2/2R^2\right]$$

+ Can solve using Lagrangian $T - V$, etc

— Gravity + Orbits

• Inverse Square Law Orbits

+ We looked at these in PHYS-3202

by saying one object is much more massive so it didn't move

+ Now we know that we can write the Lagrangian as

$$L = \tfrac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \tfrac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} + G m_1 m_2 / |\vec{r}_1 - \vec{r}_2|$$

$$= \left(\tfrac{1}{2} M \dot{\vec{R}}^2\right) + \left(\tfrac{1}{2} \mu \dot{\vec{r}}^{\,2} + G M \mu / r\right) \quad \Big\} \; m_1 m_2 = \mu M$$

+ So everything we did before still works for relative motion using fixed center of total ~~mass~~ M and orbiting object of reduced mass μ.
+ In our solar system $m_2 \gg m_1$ almost always, so $M \approx m_2$, $\mu \approx m_1$
+ Kepler's 3rd law is slightly different for each orbiting object:

$$(T/2\pi)^2 = a^3/GM$$ and M = total mass depends on m_1 a little bit even if m_2 is much bigger

+ Each object orbits the CM in an ellipse. If orbit semi-major axis is a, each has axis

$$a_1 = m_2 a / M, \quad a_2 = m_1 a / M$$

• Tidal friction

+ High tides are bulges of water due to moon's pull. But they rotate w/ earth. The pull back toward the moon slows earth's rotation
+ CM frame angular momentum $\vec{J}^* = \mu \vec{r} \times \dot{\vec{r}} + \vec{J}_\oplus + \vec{J}_m$ is conserved b/c external torques vanish on average
+ $\vec{J}_m$ very small, $\mu \vec{r} \times \dot{\vec{r}} \approx 2\pi \mu a^2 / T = \mu \sqrt{GMa}$ by Kepler's 3rd, $J_\oplus = I_\oplus \omega_\oplus$ where $I_\oplus$ = earth moment of inertia, $\omega_\oplus$ = earth rotation frequency
+ As $\omega_\oplus$ decreases, orbit axis a increases. Energy transfer to the moon's orbit!

• Restricted 3-body Problem

+ In general, it's impossible to solve for the motion of 3+ objects interacting gravitationally except on a computer. This is the <u>3-body problem</u> or <u>N-body problem</u>. Describes parts of our solar system, formation of structure in universe, etc.

+ In 3-body problem, objects + masses are primary m_1, secondary m_2, and tertiary m_3 with $m_1 \geq m_2 \geq m_3$

+ The <u>restricted 3-body problem</u> makes 3 assumptions
  a) Tertiary is very light $m_3 \ll m_2, m_1$ so its gravity does not affect the motion of m_1 or m_2
  b) The primary + secondary have circular orbits around their CM
  c) All motion is in one plane

+ Work in reference frame with primary/secondary CM rotating w/ the primary + secondary orbits. Then primary + secondary are at fixed positions $a_1\hat{x}$, $-a_2\hat{x}$.
The frame's angular velocity is $\omega\hat{z}$ where $\omega^2 = G(m_1+m_2)/(a_1+a_2)^3$

+ In this frame, the tertiary is at $\vec{r} = x\hat{x} + y\hat{y}$ by assumption. It experiences force (including fictitious)

$$\vec{F} = -m_3\left(\frac{Gm_1}{r_1^3}\vec{r}_1 + \frac{Gm_2}{r_2^3}\vec{r}_2\right) - 2m\vec{\omega}\times\dot{\vec{r}} + m\omega^2\vec{r}$$

where $\vec{r}_1 = \vec{r} - a_1\hat{x}$, $\vec{r}_2 = r + a_2\hat{x}$ are relative displacements

+ There are 5 <u>Lagrangian points</u> where $\vec{F} = 0$ for stationary tertiary. Three are on $\hat{x}$ axis:
L_1 between $m_1 + m_2$; L_2 past m_2; L_3 past m_1
Motion here is unstable: tertiary will move away.
$L_4 + L_5$ are at vertices of equilateral triangle w/ m_1, m_2
Motion here is stable due to Coriolis force