PHYS-3203 Homework 2 Due 26 Jan 2022

This homework is due to `https://uwcloud.uwinnipeg.ca/s/QGK3eGfDRgND6sC` by 10:59PM on the due date. Your file(s) must be in PDF format; they may be black-and-white scans or photographs of hardcopies (all converted to PDF), PDF prepared by LaTeX, or PDF prepared with a word processor *using an equation editor.*

1. **Geodesic on a Cone** *based on a Kibble & Berkshire problem*

   A *geodesic* is the minimal length curve on a surface between two points on that surface (or possibly in a curved space). For example, we showed that a straight line segment is a geodesic on a plane, and you may know that a great circle is a geodesic on a sphere. Here we will examine geodesics on a cone with its tip at the origin and its axis of symmetry along the z axis. The surface of the cone is at a polar angle α from the z axis.

   (a) Find the relationship between the cylindrical coordinates ρ and z on the surface of the cone and show that the distance L from point (ρ_1, φ_1) to point (ρ_2, φ_2) on the cone can be written
   $$L = \int_{\varphi_1}^{\varphi_2} d\varphi \sqrt{\rho^2 + \csc^2\alpha\, \rho'^2} \tag{1}$$
   where $\rho' = d\rho/d\varphi$. *Hint:* Use the Pythagorean theorem in cylindrical coordinates, $d\ell^2 = dz^2 + d\rho^2 + \rho^2 d\varphi^2$ and substitute for z in terms of ρ.

   (b) Show that a geodesic satisfies the equation
   $$\rho\rho'' - 2(\rho')^2 - \sin^2\alpha\, \rho^2 = 0 \ . \tag{2}$$
   Hint: Use the Euler-Lagrange equation for L.

   (c) Solve (2) for $\rho(\varphi)$ by changing variables to $\rho = 1/u$ and finding a differential equation for u. Leave your solution in terms of 2 undetermined integration constants (do not find them in terms of the boundary conditions stated above). What do the integration constants describe?

2. **Geometry with Constraints** *adapted from problems by Thornton & Marion*

   The following are constrained optimization problems in multivariable calculus. Use the method of Lagrange multipliers to solve them.

   (a) Consider a right-circular cylinder of radius r and height h. If the volume of the cylinder is fixed to V, what is the ratio r/h that minimizes the surface area?

   (b) Consider a parallelepiped circumscribed by a sphere of radius R (that is, the vertices of the parallelepiped lie on the surface of the sphere). What are the dimensions (lengths of the 3 independent sides) of such a parallelepiped of maximum volume?