Rigid Body Rotation

Math: Matrices + Tensors.

- Index Notation (most examples in 3D, but more general)
  - Suppose we choose a basis set of orthonormal vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$
    - We remember we can write any vector $\vec{a}$ as
      $$\vec{a} = a_1 \hat{\imath} + a_2 \hat{\jmath} + a_3 \hat{k}$$
    - The a_i numbers are the components of $\vec{a}$ in this basis
    - We can assemble the components in a column matrix $\vec{a} \rightarrow \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

Examples

- The dot product is $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$
  $$= \sum_i a_i b_i \equiv a_i b_i$$
  The last form is Einstein summation convention which means you sum over a repeated (paired) index
- We can define the Levi-Cività symbol such that
  $$\epsilon_{123} = 1, \qquad \epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}$$
  (minus sign every time you swap an index).
  Then the cross product is
  $$(\vec{a} \times \vec{b})_i = \sum_{jk} \epsilon_{ijk} a_j b_k$$
- Can also write the dot product as a matrix multiplication
  $$[a_1\ a_2\ a_3] \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = [\vec{a}]^T [\vec{b}]$$
  where ^{T} = transpose = flip the column into a row.

- Matrices + vectors
  - Components according to rotated unit vectors are related by a rotation matrix
    $$\vec{a} \rightarrow \begin{bmatrix} a_1' \\ a_2' \\ a_3' \end{bmatrix} = R \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

+ Other linear transformations are also matrix multiplication

+ In index notation, a matrix has an index indicating row position + one for column position

$$M_{ij} \quad \text{or} \quad M = \begin{array}{c} \text{row 1} \\ \#2 \\ \#3 \end{array}\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$
$$\qquad\qquad\qquad\qquad \text{column 1} \quad \#2 \quad \#3$$

+ Matrix multiplication is a sum of column + row numbers

$$(AB)_{ij} = \sum_k A_{ik} B_{kj} \; ; \quad (M\vec{a})_i = \sum_j M_{ij} a_j$$

+ The matrix transpose reverses row + column

$$(M^T)_{ij} \doteq M_{ji} \quad \text{If } M^T = M, \; M \text{ is } \underline{\text{symmetric}}$$

So we can see $(AB)^T = B^T A^T$

+ Rotating axes should leave dot products unchanged

$$(\vec{a} \cdot \vec{b}) \rightarrow [a_1' \; a_2' \; a_3'] \begin{bmatrix} b_1' \\ b_2' \\ b_3' \end{bmatrix} = [a_1 \; a_2 \; a_3] \, R^T R \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$\Rightarrow R^T R = \underline{I}$ Means rotation matrices are <u>orthogonal</u>

- Eigenvalues + eigenvectors

  • What happens when we multiply a square matrix on a vector?
    + Normally, the components get mixed up
    + For special vectors $\vec{a}$ given a matrix M,

$$M\vec{a} = \lambda \vec{a} \quad \text{where } \lambda \text{ is a particular number}$$

In other words, M just rescales $\vec{a}$.

  • Definition: Given a square matrix M, a <u>nonzero</u> vector $\vec{a}$

is an <u>eigenvector</u> of M if $M\vec{a} = \lambda\vec{a}$
for some number λ called <u>the eigenvalue</u>

+ Typically, an $n \times n$ matrix has n distinct eigenvectors, each with their own eigenvalue.

+ If M is (real) symmetric, eigenvectors can be made orthonormal, and eigenvalues are real.

+ So the eigenvectors of a symmetric matrix make up a basis of unit vectors

- How do we find eigenvalues and eigenvectors?

+ We can rewrite the basic equation as $(M - \lambda \mathbb{1})\vec{a} = 0$

+ $(M - \lambda\mathbb{1})$ is not invertible when λ is an eigenvalue
$\Rightarrow \det(M - \lambda\mathbb{1}) = 0$

+ This is a polynomial of order n for the eigenvalue λ called the <u>characteristic equation</u>. The n roots $\lambda_1 \dots \lambda_n$ are the n eigenvalues. They can repeat.

+ Then $M\vec{a} = \lambda_i\vec{a}$ is a system of linear eqns for the components of the eigenvector $\vec{a}$ with eigenvalue λ_i. Then do this for each eigenvalue

+ Note: You can't determine the normalization of an eigenvector from $M\vec{a} = \lambda\vec{a}$ (multiplying both sides by a scalar is the same as rescaling $\vec{a}$). So add a normalization condition

- Some notes on complex matrices + vectors

+ The dot product for complex vectors is
$\vec{a} \cdot \vec{b} = a_i^* b_i$ ← often written $(\vec{a}, \vec{b})$
b/c order matters

+ The Hermitean adjoint $A^{+} = (A^{T})^{*}$, where $*$ is the complex conjugate of each element

+ A unitary matrix has $U^{+} = U^{-1}$
  A Hermitian matrix has $H^{+} = H$

+ A Hermitian matrix has all real eigenvalues, and the eigenvectors can be chosen orthogonal

— Tensors

- Consider two vectors multiplied side by side but not dotted, ie, $\vec{a}\,\vec{b} = \overleftrightarrow{T}$

  + This has 2 indices $T_{ij} = a_i b_j$ for components w.r.t. a set of unit vectors
  + It acts on a vector by dotting
    $(\overleftrightarrow{T}\,\vec{c}) = \vec{a}\,(\vec{b}\cdot\vec{c})$ or $\sum_j a_i (b_j c_j)$

  + IT in general

- A physical object that converts a vector to another vector (linearly) is a (rank-2) tensor

  ++ These can be "tensor products" of vectors or not

  ++ One different example is the identity $\mathbb{1}$ w/components δ_{ij}

  + Given a set of axes, the components of tensors form a matrix, which acts on vectors by matrix multiplication on components $\overleftrightarrow{T}\,\vec{c} \rightarrow [T][c]$

  + So tensors have eigenvectors + eigenvalues

- How do tensor components change when axes are rotated?

  + Think about our vector product example

    For $T_{ij} = a_i b_j$,

$$T'_{ij} = \sum_{k,l} (R_{ik} a_k)(R_{jl} b_l) = R_{ik} R_{jl} T_{kl}$$

  + So the components of a tensor transform by rotating each index like a vector.

  + If we think of the components of a tensor as a matrix, we can re-organize the above

$$T'_{ij} = \sum_{k,l} R_{ik} T_{kl} (R^T)_{lj}$$

or $\overleftrightarrow{T} \rightarrow [T'] = R[T]R^T$

  + What does this mean for components of $(\overleftrightarrow{T}\vec{a})$?

$$[T'][a'] = R[T]R^T R[a] = R[T][a]$$

In other words $(\overleftrightarrow{T}\vec{a})$ is a vector, so its components rotate like a vector's components