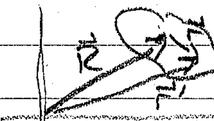
Cases of Rigid Body Motion

- General Comments
  - Motion of the body
    - Suppose we have a fixed inertial reference frame with CM of an object at $\vec{R}$



    + The position of a point on the object is
$$\vec{r}' = \vec{R} + \vec{r} \quad \text{where } \vec{r} = \text{position relative to center of mass}$$
    + The angular momentum w.r.t. the inertial origin is
$$\vec{J}' = \int dm\, \vec{r}' \times \frac{d\vec{r}'}{dt} = M\vec{R} \wedge \frac{d\vec{R}}{dt} + \vec{J} \quad \text{with } \vec{J} = \overleftrightarrow{I}_{cm}\vec{\omega}_{cm}$$
    by arguments similar to the parallel axis theorem. $\vec{J}$, $\overleftrightarrow{I}_{cm}$, $\vec{\omega}_{cm}$ are the angular momentum, inertia tensor and angular velocity around the object center of mass
    + The kinetic energy is similarly
$$T = \tfrac{1}{2} M (d\vec{R}/dt)^2 + \tfrac{1}{2}\vec{\omega}_{cm} \cdot (\overleftrightarrow{I}_{cm}\vec{\omega}_{cm})$$
    + In both cases, the motion splits into the motion of a point-like object at the CM plus body rotation

    + In the same set-up, consider the total force
$$\vec{F}'_{ext} = \int d\vec{F}'_{ext} + \int d\vec{F}'_{int} = \int dm \frac{d^2\vec{r}'}{dt^2} = \int dm \frac{d^2\vec{R}}{dt^2} = M\frac{d^2\vec{R}}{dt^2}$$
    so total external force determines CM motion b/c (1) total internal force cancels and (2) $\int dm\, \vec{r} = 0$ always

  - Torque around the origin and the CM

    + The torque around the fixed origin is
$$\vec{\tau}' = \frac{d\vec{J}'}{dt} = M\vec{R} \wedge \frac{d^2\vec{R}}{dt^2} + \frac{d\vec{J}}{dt}$$

    + But $M\vec{R} \times d^2\vec{R}/dt^2 = \vec{R} \times \vec{F}'_{ext}$.
    Therefore
$$\frac{d\vec{J}}{dt} = \int \vec{r}' \times d\vec{F}' - \vec{R} \times \int d\vec{F}'_{ext}$$

We have argued before that internal forces do not give net torque, so

$$d\vec{J}/dt = \int (\vec{r}' - \vec{R}) \times d\vec{F}'_{ext} = \int \vec{r} \times d\vec{F}'_{ext} \equiv \vec{\tau}$$

+ This tells us that the torque around the CM is given by the physical external forces. Note that uniform gravity exerts no torque around the CM (it appears to act at CM).

+ The fictitious forces in the accelerating frame w/CM at the origin are also uniform and $\propto$ mass!, so

$$\vec{\tau}_{fictitious} = -\int \vec{r} \times \left(dm\, d^2\vec{R}/dt^2\right) = -\left(dm\, \vec{r}\right) \times \frac{d^2\vec{R}}{dt^2} = 0$$

+ If we choose a more general accelerating origin (not the CM), the fictitious forces "act at the CM" and generate a torque around that origin

- Describing motion from a rotating body frame: Euler's equations

+ With respect to inertial axes, we just found

$$d\vec{J}/dt = \vec{\tau} \quad \text{(where } \vec{\tau} = \text{"external" torque)}$$

even when the origin moves with the object

+ But suppose we choose any body axes, like principal axes. These rotate with the object with angular velocity $\vec{\omega}$. In our previous notation,

$$\frac{d\vec{J}}{dt} = \dot{\vec{J}} + \vec{\omega} \times \vec{J} = \vec{\tau}$$

These are Euler's equations

+ For a rigid body, the inertia tensor w.r.t. the body does not change. In terms of the principal axes $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$, the Euler equations are

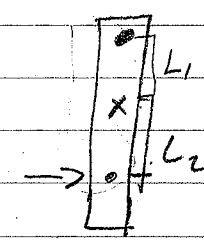
$$I_1 \dot{\omega}_1 + (I_3 - I_2)\omega_2 \omega_3 = \tau_1$$

and permutations

+ Can be difficult to use if torque is constant w.r.t fixed inertial axes.

— More examples of planar motion

• "Baseball Bat Theorem"



+ An object can pivot around a point L_1 from the center of mass. A sharp force acts on the object at a different point L_2 from the center of mass. When is the force on the pivot zero?

+ As a result of the impulse, the momentum of the object immediately after the strike is $P = M V_{cm} = M L_1 \omega$, where ω = new angular velocity.

+ Meanwhile the impulse acts with a torque creating angular momentum $P(L_1 + L_2) = I\omega$ around the pivot, where I = moment of inertia.

+ Dividing these equations, we see

$$I = M L_1^2 + M L_1 L_2$$

for no force on the pivot. Using the parallel axis theorem, $I = I_{cm} + M L_1^2$ where I_{cm} is the moment around the CM. This yields

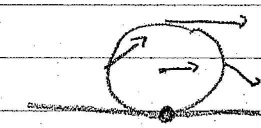
$$M L_1 L_2 = I_{cm}$$

as the condition on L_1 & L_2

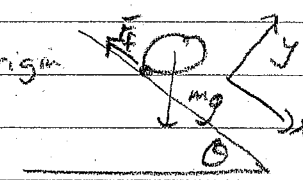
+ L_2 is the "center of percussion" for the pivot at L_1. Baseball players call it the "sweet spot" to avoid having the bat exert force on their hands.

- Rolling with and without slipping for objects with a circular profile

  + Rolling is translational + rotational motion. + Without slipping, the contact point is always instantaneously at rest w.r.t. the surface. This means the linear velocity $v = \omega R$, where ω is the angular velocity of the rotation + R the radius of the circular profile. There is static friction.



  + Ex: Rolling on a slope, center of mass as origin. Linear motion is determined by



$$M\ddot{x} = Mg\sin\theta - \mu_s Mg\cos\theta \quad \text{as usual.}$$

Meanwhile, in center of mass frame, torque from friction:
So $$I_{cm}\dot{\omega} = \mu_s Mg\cos\theta\, R$$
(assuming the axis of rotation is principal)
But then $\omega = \dot{x}/R$, so
$$\ddot{x} = \frac{g\sin\theta}{1 + I_{cm}/MR^2}$$

  + Same thing, using the contact point as origin. This time, the torque is
$$I\dot{\omega} = Rmg\sin\theta \quad w/\ I = I_{cm} + MR^2$$
Again, where x = center of mass position, $\dot{\omega} = \ddot{x}/R$
We get the same answer.

  + Suppose the center of mass moves a distance x on the incline. What's the speed? Energy conservation:

$$mgx\sin\theta = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}I_{cm}(v/R)^2$$

+ With slipping, there is kinetic friction: $v \neq \omega R$
Consider the same example.

$$\ddot{x} = g\sin\theta - \mu_k g\cos\theta$$

$$\dot{\omega} = \mu_k M g R\cos\theta / I_{cm}$$

So ω and $v = \dot{x}$ have a constant ratio

– Force-free Motion = No torques

• Stability of rotation around a principal axis

+ Suppose the initial motion of a freely-rotating object is given by $\vec{\omega} = \omega_1\hat{e}_1 + \omega_2\hat{e}_2 + \omega_3\hat{e}_3$ with $\omega_1, \omega_2 \ll \omega_3$ where $\hat{e}_i$ are principal axes. Rotation mostly around $\hat{e}_3$

+ Since ω_1 + ω_2 are small, neglect $\omega_1\omega_2$ in Euler equations. They become

$$I_3\dot{\omega}_3 \approx 0, \quad I_1\dot{\omega}_1 = (I_2 - I_3)\omega_3\omega_2, \quad I_2\dot{\omega}_2 = (I_3 - I_1)\omega_1\omega_3$$

+ So ω_3 is basically constant. By differentiating,

$$\ddot{\omega}_1 \approx \frac{(I_3 - I_1)(I_2 - I_3)}{I_1 I_2}\omega_3^2\omega_1$$

We guess a (complex) solution $\omega_1 = Ae^{i\Omega t}$
We set

$$\Omega^2 = \frac{(I_3 - I_1)(I_3 - I_2)}{I_1 I_2}\omega_3^2$$

and ω_2 takes the same functional form.

+ If $I_3 > I_1, I_3 > I_2$ or $I_3 < I_1, I_3 < I_2$ (ie, largest or smallest moment), Ω is real.
That means ω_1 + ω_2 oscillate + stay small.
If $I_1 < I_3 < I_2$ or $I_2 < I_3 < I_1$ (I_3 is middle moment), Ω is imaginary, so ω_1 + ω_2 grow.

+ In other words, rotation around the axis with largest or smallest is stable. The middle one is unstable.
If 2 moments have equal moments, rotation around them is unstable, but the other is stable (check).

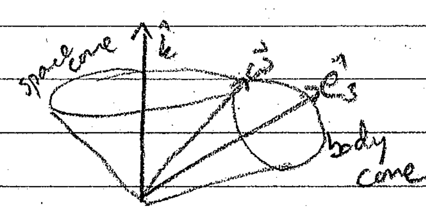
- Free Rotation of a Symmetric Top (Rotating Object)
  - Consider a symmetric rigid body (top) with principal moments $I \equiv I_1 = I_2$ and I_3. It has angular velocity $\vec{\omega} = \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3$ in terms of principal axes

  - The Euler equations are
  $$I_3 \dot{\omega}_3 = 0, \quad I\dot{\omega}_1 = (I - I_3)\omega_3 \omega_2, \quad I\dot{\omega}_2 = (I_3 - I)\omega_3 \omega_1$$
  we immediately see $\omega_3 = \text{const}$

  - We can define $\Omega = (I_3 - I)\omega_3 / I$, so
  $$\dot{\omega}_1 + \Omega \omega_2 = 0, \quad \dot{\omega}_2 - \Omega \omega_1 = 0$$
  - We've seen this type of equation before (Lorentz force)
  Answer is
  $$\omega_1 = A\cos(\Omega t + \delta), \quad \omega_2 = A\sin(\Omega t + \delta)$$

  - Describe this motion: ω_1 & ω_2 describe circular motion in the plane of $\hat{e}_1, \hat{e}_2$. Further, the magnitude is constant. This means $\vec{\omega}$ precesses around $\hat{e}_3$ with frequency Ω. $\vec{\omega}$ traces a cone called the body cone around $\hat{e}_3$

  - Alternately, note that $\vec{J}$ is conserved. Choose inertial axes so $\vec{J} = J\hat{k}$. Further, the conserved kinetic energy $T = \frac{1}{2}\vec{\omega}\cdot(\overleftrightarrow{I}\vec{\omega}) = \frac{1}{2}\vec{\omega}\cdot\vec{J}$, so the angle between is constant. $\Rightarrow \vec{\omega}$ precesses around $\hat{k}$ also. Then $\vec{\omega}$ traces out a cone called the space cone around $\hat{k}$

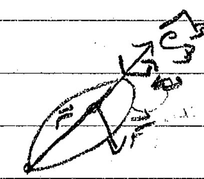


  - The earth's rotation is actually not lined up with its principal axis, so it precesses like this. For the earth, $I \approx I_3$, so Ω is small, leading to a ~300 day period. It's actually longer b/c earth is not quite rigid

— Precession from a Small Torque

- Consider a body rotating around an axis with 1 fixed point but otherwise free to rotate

  + To avoid free-body precession, assume the axis is principal axis $\hat{e}_3$. Further assume a small force $\vec{F}$ acts at point $\vec{r}$ on the axis.



  + Assuming the force is small, the motion of the axis will be slow compared to the rotation around the axis. $\Rightarrow$ We can treat the motion as a slow change in direction of $\hat{e}_3$ (and $\vec{\omega}$) and not rotation around $\hat{e}_1$ or $\hat{e}_2$

  + The equation of motion is then

$$\frac{d\vec{J}}{dt} \approx I_3 \frac{d\vec{\omega}}{dt} = \vec{r} \times \vec{F}$$

    Because $\vec{r} \parallel \vec{\omega}$, $\vec{\omega}$ changes direction only, $\perp$ to $\vec{F}$!

  + Suppose the force is gravity $-Mg\hat{k}$. The spinning object is a top or gyroscope. If the center of mass is a distance R from the fixed point, $\vec{\omega} = \omega \hat{e}_3$, $\vec{r} = R\hat{e}_3$, so

$$I_3 \omega \frac{d\hat{e}_3}{dt} = -MgR\, \hat{e}_3 \wedge \hat{k} \Rightarrow \frac{d\hat{e}_3}{dt} = \vec{\Omega} \wedge \hat{e}_3$$

    where $\vec{\Omega} = (MgR/I_3\omega)\,\hat{k}$.
    This is again precession around $\hat{k}$ at frequency Ω.

  + Note that $\Omega \propto (I_3 \omega)^{-1}$. Rapidly spinning, wide objects barely precess. This is why gyroscopes point in a fixed direction

  + The sun + moon exert a torque on the earth due to its slightly oblate shape. This causes precession of the equinoxes with period $\simeq$ 26000 years.