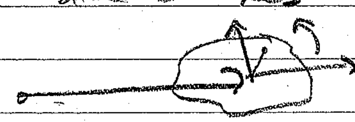
Moments of Inertia for Rigid Bodies

– Rotation Around a Fixed Axis

- A *rigid body* is an object of fixed shape. In other words, it's possible to find a (possibly accelerating) coordinate system where all pieces of the object are at rest.



- Suppose a rigid object rotates around a fixed axis, ie, there is a post or something it rotates around.

  + Use cylindrical coordinates w/ rotation around z axis

  + Then any piece of the object remains at fixed z, ρ and moves only in φ with $\omega = d\varphi/dt$

  + The angular momentum along the axis is

$dm = d^3\vec{x}\,\rho(\vec{x})$ where $\rho(\vec{x})$ = density = small box of mass

$$J_z = \int dm\, \rho\, v_\varphi = \left(\int dm\, \rho^2\right)\omega \equiv I\omega \qquad \text{b/c } v_\varphi = \rho\frac{d\varphi}{dt}$$

where I is the moment of inertia around z. It's a property of the object.

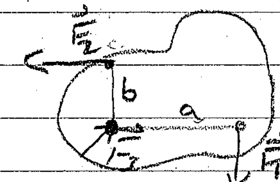
  + The kinetic energy also depends on I

$$T = \frac{1}{2}\int dm\, v_\varphi^2 = \frac{1}{2}\left(\int dm\,\rho^2\right)\omega^2 = \frac{1}{2}I\omega^2$$

  + Recall that $d\vec{J}/dt = \vec{\tau}_{ext}$. That is, internal torques cancel (as if all internal forces are central). This means rigid bodies do not start to rotate spontaneously.

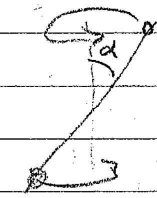
  + So a stationary object (in equilibrium) has no net external force (it is not accelerating) and no net external torque

- Equilibrium example: An object is fixed to some axis. A force $\vec{F}_1$ acts in the $-\hat{j}$ direction at a point $x=a, y=0$. Another, $\vec{F}_2$, acts in the $\hat{i}$ direction at $x=0, y=b$.



  + The total external torque is
  $$\vec{\tau} = (F_2 b - F_1 a)\hat{k} \to 0 \text{ if stationary}$$
  + Meanwhile, the axis must exert a force
  $$\vec{F}_3 = -\vec{F}_1 - \vec{F}_2 = F_2\hat{i} + F_1\hat{j}$$
  for equilibrium

- A More Surprising Example:



  + Consider 2 masses m each at opposite ends of a light rod of length l at an angle α from z axis rotating around z

  + Kinetic energy is $T = \frac{1}{2}m\left(\frac{l}{2}\sin\alpha\,\omega\right)^2 + \frac{m}{2}\left(\frac{l}{2}\sin\alpha\,\omega\right)^2$
  $$= \tfrac{1}{4}ml^2\sin^2\alpha\,\omega^2$$
  as you might expect

  + At orientation φ, the mass positions are
  $$\vec{r} = \pm(\sin\alpha\cos\varphi\,\hat{i} + \sin\alpha\sin\varphi\,\hat{j} + \cos\alpha\,\hat{k})\frac{l}{2}$$
  and velocities $\vec{v} = \pm\,\omega\sin\alpha\,(-\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j})\,l/2$
  So
  $$\vec{J} = m\omega l^2/2\left[-\cos\alpha\sin\alpha(\sin\varphi\,\hat{j} + \cos\varphi\,\hat{i}) + \sin^2\alpha\,\hat{k}\right]$$

  + What's with the extra components?
  Note that (even with ω constant), there is a torque
  $$\vec{\tau} = d\vec{J}/dt = (m\omega^2 l^2/2)\cos\alpha\sin\alpha(\sin\varphi\,\hat{i} - \cos\varphi\,\hat{j})$$
  from the post through the axis

- Moments of a mass distribution:

A moment of a distribution is a measure of the shape by taking powers of position

- 0^{th} Moment = total mass

$$M = \int dm = \int d^3\vec{r}\, \rho(\vec{r})$$

- 1^{st} Moment = center of mass position (weighted avg position)

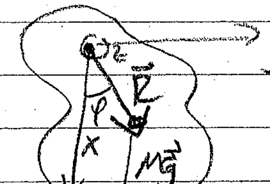
  + $$M\vec{R} = \int dm\, \vec{r} = \int d^3\vec{r}\, \rho(\vec{r})\, \vec{r}$$

  + How does gravity act? $d\vec{F} = dm\,\vec{g}$ is uniform, so the torque is

$$\vec{\tau} = \int \vec{r} \times d\vec{F} = \int dm\, \vec{r} \times \vec{g} = M\vec{R} \times \vec{g}$$

In other words, the torque due to gravity is as if the gravitational force of the entire object acts at the center of mass

  + Example: compound pendulum



This is any rigid object hanging from a point.

Assuming it's fixed, $\vec{J} \equiv I\, d\varphi/dt\, \hat{k}$ w/ I as before and torque $\vec{\tau} = -MRg \sin\varphi\, \hat{z}$

With I = constant,

$$I\, d^2\varphi/dt^2 = -MRg\sin\varphi \approx -MRg\,\varphi \quad \text{@ small angle.}$$

This is a simple pendulum of length $l = I/MR$.

- 2^{nd} Moment: Inertia Tensor

  + Consider a rigid body rotating around some origin. Inertial axes are given by fixed $\hat{\imath}, \hat{\jmath}, \hat{k}$ unit vectors. There are rotating body axes $\hat{x}, \hat{y}, \hat{z}$.

Any point in the body

+ Any point in the object is fixed w.r.t the body axes. Therefore,
$$\frac{d\vec{r}}{dt} = \vec{\omega}\times\vec{r} \quad \text{since} \quad \dot{\vec{r}}=0$$
where $d\vec{r}/dt$ is wrt $\hat{i},\hat{j},\hat{k}$ axes, $\dot{\vec{r}}$ wrt $\hat{x},\hat{y},\hat{z}$.
Note that we're ignoring movement of the origin for now

+ The angular momentum is
$$\vec{J} = \int dm\, \vec{r}\times(\vec{\omega}\times\vec{r}) = \int dm \left[|\vec{r}|^2\vec{\omega} - (\vec{r}\cdot\vec{\omega})\vec{r}\right]$$
and kinetic energy is

triple product
$$T = \frac{1}{2}\int dm\, (\vec{\omega}\times\vec{r})^2 = \frac{1}{2}\int dm\, \vec{\omega}\cdot(\vec{r}\times(\vec{\omega}\times\vec{r}))$$
$$= \frac{1}{2}\int dm \left[r^2\omega^2 - (\vec{r}\cdot\vec{\omega})^2\right]$$

+ These can be written in terms of an inertia tensor $\overleftrightarrow{I}$ with components
$$I_{ij} = \int dm \left[r^2\delta_{ij} - r_i r_j\right]$$
so $\vec{J} = \overleftrightarrow{I}\vec{\omega}$, $T = \frac{1}{2}\vec{\omega}\cdot(\overleftrightarrow{I}\vec{\omega})$

I_{ij} components can be calculated w.r.t any set of axes (inertial or body axes)

+ Note that $\overleftrightarrow{I}$ also changes depending on the choice of origin.

- Properties of the Inertia Tensor
  - Principal axes + principal moments

+ Nonzero products of inertia complicate how we evaluate angular momentum and kinetic energy. We can choose axes to make them vanish. These are principal axes.

+ Because they are determined by the object's geometry, principal axes are rotating body axes

+ Call the unit vectors of principal axes $\hat{e}_1, \hat{e}_2, \hat{e}_3$. With respect to these unit vectors,

$$\overleftrightarrow{I} \rightarrow \begin{bmatrix} I_1 & & 0 \\ & I_2 & \\ 0 & & I_3 \end{bmatrix} \qquad I_{1,2,3} = \text{principal moments}$$

so angular velocity $\vec{\omega}$ along a principal axis is an eigenvector of $\overleftrightarrow{I}$.

+ To find principal axes + moments: Construct $\overleftrightarrow{I}$ as a matrix w.r.t. any body axes w/unit vectors $\hat{x}, \hat{y}, \hat{z}$. The principal axis vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are the normalized eigenvectors. The principal moments are the eigenvalues.

+ Sometimes, we can determine principal axes by symmetry (makes things easy). If we can find one by symmetry, call it $\hat{e}_3$. To find $\hat{e}_1, \hat{e}_2$, pick any body axes $\hat{x}, \hat{y}$ with $\hat{z} = \hat{e}_3$. Then

$$\overleftrightarrow{I} \rightarrow \begin{bmatrix} I_{xx} & I_{xy} & 0 \\ I_{xy} & I_{yy} & 0 \\ 0 & 0 & I_{zz} = I_3 \end{bmatrix}$$

To find $\hat{e}_1, \hat{e}_2$ and I_1, I_2, you can just find eigenvalues + eigenvectors of the 2x2 block.

+ It should be clear that $\overleftrightarrow{I}$ is symmetric as a matrix, so principal axes can be chosen orthogonal.

+ A symmetric object has 2 or more principal moments equal.

+ If $\vec{\omega} = \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3$, then

$$\vec{J} = I_1 \omega_1 \hat{e}_1 + I_2 \omega_2 \hat{e}_2 + I_3 \omega_3 \hat{e}_3$$

$$T = \frac{1}{2}\left(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2 \right)$$

- Parallel Axis Theorem (Steiner)
  - A natural origin for body axes is the center of mass. But sometimes the rotational axis is through another origin
  - Suppose the center of mass is at position $\vec{R}$ compared to the other origin. Consider the inertia tensor $\overleftrightarrow{I}'$ around the new origin using the same unit vectors $\hat{x}, \hat{y}, \hat{z}$, so the axes through the 2 origins are parallel.
  - The position of a point in the body relative to the new origin is

$$\vec{r}\,' = \vec{R} + \vec{r}$$

where $\vec{r}$ is the position relative to the center of mass

  - Therefore, the inertia tensor is

$$I'_{ij} = \int dm \left[(\vec{R}+\vec{r})^2 \delta_{ij} - (R+r)_i (R+r)_j \right]$$

$$= \int dm \left(\vec{r}^{\,2} \delta_{ij} - r_i r_j \right) + \int dm \left(\vec{R}^2 \delta_{ij} - R_i R_j \right)$$

$$+ \int dm \left(2\vec{R}\cdot\vec{r}\,\delta_{ij} - R_i r_j - r_i R_j \right)$$

  - The 1st term is I_{ij}, the inertia tensor around the center of mass. Meanwhile, $\vec{R}$ is a constant, so it factors out, so the last line has integrals of the form

$$\int dm \, r_i = 0$$ b/c the center of mass in $\vec{r}$ coordinates is at zero.

  - Therefore in total

$$I'_{ij} = I_{ij} + M\left(R^2 \delta_{ij} - R_i R_j \right)$$