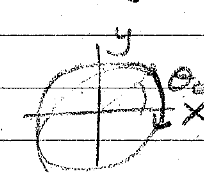
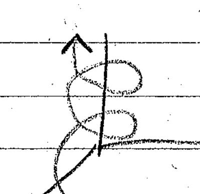
Noninertial Reference Frames

General Principles

- Preview: Lorentz force / Particle in Magnetic Field
  - The force on a charged particle is
    $$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B}) \quad \text{for charge } q, \ \vec{E} + \vec{B} \text{ fields}$$
  - Let's consider $\vec{E} = 0$, $\vec{B} = B\hat{k}$
    + Because $\vec{F}\cdot\vec{v} = 0$, the Lorentz force does no work (conservative in an odd way)
    + Equations of motion are
      $$m\ddot{x} = qB\dot{y}, \quad m\ddot{y} = -qB\dot{x}, \quad m\ddot{z} = 0$$
    + For reasons we'll see below, define $\omega_B = qB/m$
  - Solution by coupled ODEs
    + First, $\dot{z} = \text{const} = v_z$
    + Remaining
      $$\ddot{x} = \omega_B\dot{y}, \quad \ddot{y} = -\omega_B\dot{x}$$
    + If we integrate, we see $\dot{y} = -\omega_B x + \omega_B x_0$ for some integration constant x_0
    + Then
      $$\ddot{x} + \omega_B^2 x = \omega_B^2 x_0 \Rightarrow x = x_0 + r\cos(\omega_B t + \theta_0)$$
      with r + θ_0 constants
    + Plugging back
      $$\dot{y} = -r\omega_B\cos(\omega_B t + \theta_0) \Rightarrow y = y_0 - r\sin(\omega_B t + \theta_0)$$
      constant y_0
    + Path is a helix:
      Circular motion in xy plane (clockwise)
      Constant velocity in z
      Speed in xy plane constant $= v_\perp$ ← $\perp$ to $\vec{B}$
      and radius is $r = v_\perp/\omega_B$





ω_B = cyclotron frequency

- Solution by complex numbers
  - Start again with

    $$\ddot{x} = \omega_B \dot{y}, \quad \ddot{y} = -\omega_B \dot{x}$$

  - Define $u = x + iy \Rightarrow \ddot{u} = -i\omega_B \dot{u}$
  - This has solution

    $$\dot{u} = -iv_\perp e^{-i(\omega_B t+\theta_0)} \Rightarrow u = u_0 + (v_\perp/\omega_B)\, e^{-i(\omega_B t+\theta_0)}$$

    The $-i$ is chosen so position amplitude is real.
  - This breaks down to

    $$x = x_0 + (v_\perp/\omega_B)\cos(\omega_B t+\theta_0), \quad y = y_0 - (v_\perp/\omega_B)\sin(\omega_B t+\theta_0)$$

    as before.

- What if we used a rotating (noninertial) set of axes?
  - We can take them rotating at ω_B clockwise

    $$\vec{\omega} = -\omega_B \hat{k}$$

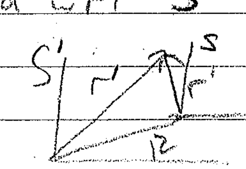
  - In this frame, particle is still in xy plane, constant velocity in z
  - Compared to Newton's laws, looks like no force
    How does this relate to EOM

    $$d\vec{v}/dt = \vec{\omega} \times \vec{v} \quad ?$$

– Acceleration of origin (with fixed axis orientation)

- Suppose frame S' is inertial and the origin of S has position $\vec{R}(t)$ measured w.r.t. S'
  - Axes of S' always ∥ to axes of S (assumed)
  - The position $\vec{r}$ of an object measured wrt S is given by

    $$\vec{r} = \vec{r}\,' - \vec{R}$$

    where $\vec{r}\,'$ is measured wrt S'
  - So

    $$\vec{v} = d\vec{r}/dt = \vec{v}\,' - d\vec{R}/dt$$

    ie, there is a shift of velocity
    This is true even if S & S' are inertial (unaccelerating)

+ But if S accelerates,
$\vec{a} = \vec{a}' - d^2\vec{R}/dt^2 \Leftarrow$ acceleration differs!

∘ What about the force measured in the accelerated frame?

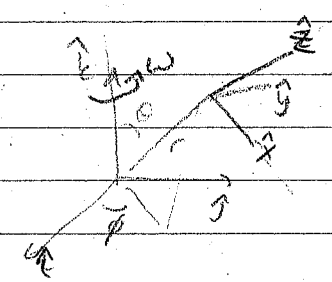
+ "Newton's 2nd Law" says $\vec{F}' = \vec{F}_{phys} - m\,d^2\vec{R}/dt^2$
The 2nd term on the RHS is a "fictitious force"

+ Example: Consider a ball or stationary pendulum in accelerating car, bus, elevator, etc.
A ball will seem to accelerate "out of nowhere" due to the fictitious force as seen by accelerated observer

+ If a frame is uniformly + constantly accelerated, a freely moving object appears to be accelerated as if by uniform gravity.
The idea that the gravitational force is fictitious is the equivalence principle + a key idea of general relativity.

– Rotating reference frame

• Setup:
+ There is some set of fixed (inertial) axes given by unit



+ We want to work with x, y, z axes with origin at spherical coordinates (r, θ, ϕ). These axes are rotating around the $\theta = 0$ $\hat{k}$ axis at angular frequency ω, ie, $\vec{\omega} = \omega\hat{k}$.

+ To be clear about which axes are which, call the rotating unit vectors on the x, y, z axes $\hat{x}$, $\hat{y}$, and $\hat{z}$.

- We want to understand how $\hat{x}, \hat{y}, \hat{z}$ change in time.
  - + Consider the case where $\hat{x} = \hat{\theta}$, $\hat{y} = \hat{\phi}$, $\hat{z} = \hat{r}$ at the position (r, θ, ϕ)
  - + The rotation around $\hat{k}$ means $\dot{\phi} = \omega$
  - + From a previous homework, we know

$$\hat{x} = \hat{\theta} = \cos\theta(\cos\phi\,\hat{i} + \sin\phi\,\hat{j}) - \sin\theta\,\hat{k}$$
$$\hat{y} = \hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$
$$\hat{z} = \hat{r} = \sin\theta(\cos\phi\,\hat{i} + \sin\phi\,\hat{j}) + \cos\theta\,\hat{k}$$

  - + With $\dot{\theta} = 0$, $\dot{\phi} = \omega$, we also know from HW

$$\frac{d\hat{x}}{dt} = \frac{d\hat{\theta}}{dt} = \omega\cos\theta\,\hat{y} = \vec{\omega}\times\hat{x}$$
$$d\hat{y}/dt = \frac{d\hat{\phi}}{dt} = -\omega(\cos\theta\,\hat{x} + \sin\theta\,\hat{z}) = \vec{\omega}\times\hat{y}$$
$$d\hat{z}/dt = \frac{d\hat{r}}{dt} = \omega\sin\theta\,\hat{y} = \hat{\omega}\times\hat{z}$$

- We can use this to determine derivatives of any vector for a rotation
  - + Note that we can write any vector $\vec{a}$ as
$$\vec{a} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$$
at any time
  - + The time derivative is

previously, we assumed unit vectors constant.

$$\frac{d\vec{a}}{dt} = \left(\dot{a}_x\hat{x} + a_x(d\hat{x}/dt)\right) + \dots \equiv \mathring{\vec{a}} + \vec{\omega}\times\vec{a}$$

  - + We use KB notation to mean $\mathring{\vec{a}}$ is given by derivatives of the components w.r.t. the rotating axes. Meanwhile $d\vec{a}/dt$ is the full derivative including the rotation of $\hat{x}, \hat{y}, \hat{z}$ axes.

+ This also means a different set of axes $\hat{x}', \hat{y}', \hat{z}'$ rotating in the same way has

$$\frac{d\hat{x}'}{dt} = \vec{\omega} \times \hat{x}', \quad \frac{d\hat{y}'}{dt} = \vec{\omega} \times \hat{y}', \quad \frac{d\hat{z}'}{dt} = \vec{\omega} \times \hat{z}'$$

b/c the components wrt $\hat{x}, \hat{y}, \hat{z}$ are constant.

+ Note that this just accounts for rotation + Not motion of the (x,y,z) origin.

+ Because $\vec{\omega} \times \vec{\omega} = 0$, $d\vec{\omega}/dt = \dot{\vec{\omega}}$. The change of the angular velocity is the same in the rotating frame as in an inertial frame.

- Acceleration / Fictitious Forces in Noninertial Frames.

  - Relation of velocity between inertial + noninertial frame

    + Our rotating axes above also have a moving origin located at $\vec{R}(t)$ w.r.t. the inertial frame origin.

    + The inertial frame velocity is

$$\vec{v}' = \frac{d\vec{r}'}{dt} = \frac{d\vec{r}}{dt} + \frac{d\vec{R}}{dt} = \frac{d\vec{R}}{dt} + \dot{\vec{r}} + \vec{\omega} \times \vec{r}'$$

    + The inertial frame acceleration is

$$\vec{a}' = \frac{d^2\vec{r}'}{dt^2} = \frac{d^2\vec{R}}{dt^2} + \ddot{\vec{r}} + \vec{\omega} \times \dot{\vec{r}} + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \dot{\vec{r}} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$= \frac{d^2\vec{R}}{dt^2} + \ddot{\vec{r}} + \dot{\vec{\omega}} \times \vec{r} + 2\vec{\omega} \times \dot{\vec{r}} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

    + If the motion of $\vec{R}$ is due to the same rotation around the origin,

$$\frac{d\vec{R}}{dt} = \vec{\omega} \times \vec{R}, \quad \frac{d^2\vec{R}}{dt^2} = \dot{\vec{\omega}} \times \vec{R} + \vec{\omega} \times (\vec{\omega} \times \vec{R})$$

- The rotating observer sees a variety of fictitious forces

  + $\vec{F} \equiv m\ddot{\vec{r}} = \vec{F}'_{(phys)} - m\dot{\vec{\omega}} \times (\vec{R}+\vec{r}) - 2m\vec{\omega}\times\dot{\vec{r}}$
  $- m\vec{\omega}\times(\vec{\omega}\times(\vec{R}+\vec{r}))$

  + $\vec{F}'_{(phys)}$ is the actual force
  + $-m\dot{\vec{\omega}}\times(\vec{R}+\vec{r})$ is called the <u>transverse</u> <u>force</u>

  + $-2m\vec{\omega}\times\dot{\vec{r}}$ is the <u>Coriolis</u> <u>force</u>

  + $-m\vec{\omega}\times(\vec{\omega}\times(\vec{R}+\vec{r})) = -m(\vec{\omega}\cdot(\vec{R}+\vec{r}))\vec{\omega} + m\omega^2(\vec{R}+\vec{r})$
    is the <u>centrifugal</u> <u>force</u>

- Back to our magnetic field example, we can choose $|\vec{R}| = v_\perp/\omega_B$, $\vec{r} = 0$.

  + Inertial frame velocity is $\vec{v}' = d\vec{R}/dt = \vec{\omega}\times\vec{R}$
    so Lorentz force with $\vec{\omega} = -\omega_B\hat{k} = -q\vec{B}/m$
    is
    $$\vec{F}' = q\vec{v}'\times\vec{B} = +m\vec{\omega}\times\vec{v}' = +m\vec{\omega}\times(\vec{\omega}\times\vec{R})$$

  + The only fictitious force is centrifugal $-m\vec{\omega}\times(\vec{\omega}\times\vec{R})$
    They exactly cancel!

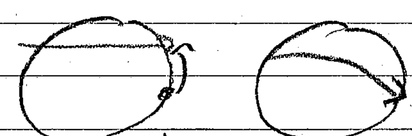
- <u>Example</u>: Puck on merry-go-round. The puck can move in the $z=0$ plane on a disk rotating with $\vec{\omega} = \omega\hat{z}$ (constant). Let's compare inertial and rotating points of view
  - \+ From an inertial frame, the puck moves in a straight line (once it starts moving). We may allow friction.
  - \+ In the merry-go-round rotating frame, the apparent force includes centrifugal + Coriolis (origin at merry-go-round axis stationary)

$$\vec{F} = \vec{F}_{friction} + m\omega^2 \vec{r} - 2m\vec{\omega}\times\dot{\vec{r}}$$

simplified

Centrifugal = pushes out, Coriolis = push of velocity to right

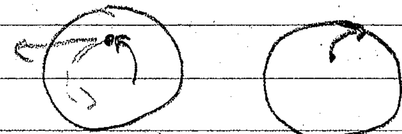
* An inertial observer throws the puck across the turntable



inertial　　rotating

In inertial frame, exit point "meets" puck. Rotating observer sees puck bend right

\+ A rotating observer lets go of the puck



inertial　　rotating

In the rotating frame, the puck moves outwards + right.

\+ The puck is sitting at a fixed location on the merry-go-round? What's the minimum coefficient of static friction?

Circular motion

In inertial frame $-m\omega^2\rho\,\hat{\rho} = m\vec{a}_{centrip} = \vec{F}_{friction}$, $|\vec{F}| \le \mu_s mg$

$\Rightarrow \mu_s \geq \omega^2\rho/g$

ie, static friction provides centripetal acceleration.

At rest in rotating frame $\vec{F}_{friction} + m\omega^2\rho\hat{\rho} = 0$

Same result from balancing friction vs centrifugal.