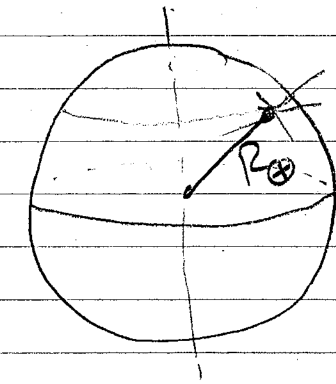
Effects on the Earth

- Our reference frame on the Earth's surface



• As above, we choose an origin at $\vec{R}$ given by $R = R_\oplus$, polar + azimuthal angles θ, ϕ

+ ϕ can be chosen as longitude (measured from prime meridian)

+ θ is colatitude. Latitude λ is polar angle measured from equator $\lambda = \pi/2 - \theta$.

+ There are fixed axes $\hat{x}, \hat{y}, \hat{z}$ at center of Earth. $\hat{z}$ points north.

• Earth rotates with angular velocity $\vec{\omega} = \omega \hat{z}$ where $\omega = 2\pi/(1\,\text{day})$

+ The way we defined θ, ϕ as colatitude + longitude means $\hat{r}, \hat{\theta}, \hat{\phi}$ axes are rotating

+ We can also set $\hat{x}, \hat{y}, \hat{z}$ axes with origin at $\vec{R}$ (rotating). Commonly, $\hat{z}$ will be vertical, $\hat{x} + \hat{y}$ along east + north.

• The force on an object as seen from a rotating frame is

$$\vec{F}(\text{phys}) - 2m\vec{\omega} \times \dot{\vec{r}} - m\vec{\omega} \times (\vec{\omega} \times (\vec{R} + \vec{r}))$$

+ We are ignoring the transverse force since $|\dot{\vec{\omega}}| \ll \omega^2$

+ For motion near the surface of the earth, ie, where $\vec{F}_{grav} \simeq m\vec{g}$, centrifugal force $\simeq -m\vec{\omega} \times (\vec{\omega} \times \vec{R})$.

— Apparent gravity: which way is up?

- The force (real + fictitious) on a stationary object is

$$\vec{F} = -mg\hat{r} + m\omega^2 R_\oplus \hat{r} - m(\vec{\omega}\cdot\vec{R})\vec{\omega}$$

+ The force of gravity is $-GMm/|\vec{R}+\vec{r}|^2\,\hat{r} \simeq -\left(\frac{GM}{R_\oplus^2}\right)m\hat{r}$ so "gravitational acceleration $g \equiv GM/R_\oplus^2$

+ Using $\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$, the centrifugal force is

$$m\omega^2 R_\oplus[(1-\cos^2\theta)\hat{r} + \cos\theta\sin\theta\,\hat{\theta}]$$
$$= m\omega^2 R_\oplus(\sin^2\theta\,\hat{r} + \tfrac{1}{2}\sin(2\theta)\,\hat{\theta})$$

- A plumb line is hung to determine vertical (this is basically a pendulum at equilibrium).

+ It hangs in the direction of the force, at an angle α from $\hat{r}$ where

$$\tan\alpha = \frac{\omega^2 R_\oplus \sin(2\theta)/2}{g - \omega^2 R_\oplus^2 \sin^2\theta} \simeq \frac{\omega^2 R_\oplus}{2g}\sin(2\theta) \simeq \alpha$$

This is about 6 arc-min max (1 arc-min $= \frac{1}{60}$ degree), so small angle approx. is justified

+ Angle is toward the equator from $\hat{r}$
+ We will take the plumb line as the direction of $\hat{z}$, our rotating Cartesian axis

- We can therefore define an apparent gravitational acceleration

$$\vec{g}_{eff} = \vec{g} + \vec{\omega}\times(\vec{\omega}\times\vec{R})$$

+ The direction is $-\hat{z}$ (ie downward vertical)
+ Magnitude is given by

$$g_{eff}^2 = (g - \omega^2 R_\oplus \sin^2\theta)^2 + \left(\frac{\omega^2 R}{2}\right)^2 \sin^2(2\theta)$$

or $g_{eff} \simeq g - \omega^2 R_\oplus \sin^2\theta$

+ At the pole, $g_{eff} = g$. At the equator, $g_{eff} = g - \omega^2 R_\oplus$. The difference is $\simeq 34\ \mathrm{mm/s^2}$ or a few % of g

+ Use $\vec{g}_{eff}$ for $\vec{g}$ in these notes. For convenience, we also shift colatitude θ so $\hat{k}\cdot\hat{z} = \cos\theta$.

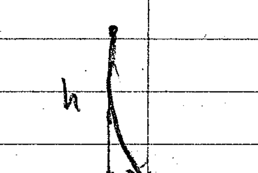
- In reality:
  + The earth's surface is roughly $\perp$ $\vec{g}_{eff}$. So it is oblate, not spherical. Poles are closer to center, so g_{eff} is even slightly larger. (maybe more later)
  + Other deformations of Earth, like mountains, change $\vec{g}_{eff}$ locally. + altitude

  + The standard value $g \simeq 9.806$ m/s^2 is agreed upon by g_{eff} at one location theoretically adjusted to sea level + latitude $45°$.
  + Ranges from ~ 9.78 m/s^2 to ~ 9.83 m/s^2.

— Coriolis Force Effects (by examples)

- Trade Winds: Warm air rises near equator + heads to poles. Cool air flows toward equator. The Coriolis Force deflects it west, leading to trade winds. (At higher latitudes, other meteorological effects push winds the other direction — the case in most of US + Canada)

- Freely falling object



  + Take $\hat{z}$ vertical, $\hat{x}$ east, $\hat{y}$ north
  + An object falls from $x=0$, $y=0$, $z=h$. Where does it land? The Coriolis Force bends the path from vertical.

  + We proceed perturbatively meaning we can treat Coriolis as a small effect. So we do not consider Coriolis forces due to velocity changes from Coriolis

  + Without the Coriolis force, $x=y=0$, $z=h-\frac{1}{2}gt^2$. Therefore, the Coriolis force is

$$-2m\vec{\omega}\times\dot{\vec{r}} \simeq -2m\omega\dot{z}\sin\theta\,\hat{x} = +2m\omega g t\sin\theta\,\hat{x}$$

$$(\vec{\omega} = \omega(\sin\theta\,\hat{y} + \cos\theta\,\hat{z}))$$

+ We therefore have $\ddot{y} = 0$, so there is no motion north, and

$$\ddot{x} = 2\omega g t \sin\theta \Rightarrow \dot{x} = \omega g t^2 \sin\theta$$
$$\Rightarrow x = \tfrac{1}{3}\omega g t^3 \sin\theta$$

+ Finally, we know it hits ground at $t = \sqrt{2h/g}$, so

$$x = \tfrac{1}{3}\omega\sqrt{8h^3/g}\,\sin\theta.$$

At $45°$ latitude, this is ≈ 16 mm for $h = 100$m (small!)

+ Note: by definition, the apparent gravitational force is always toward $-\hat{z}$.

+ This is purely due to rotation, not the curvature of the earth, etc. It is independent of $R_\oplus$ as long as $h \ll R_\oplus$

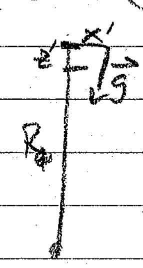
• (Counter-) Example: Freely falling object in inertial frame.

+ Set up inertial unit vectors $\hat{x}', \hat{y}', \hat{z}'$ with the object initially at $x' = y' = 0$, $z' = h$ and moving with the earth, i.e. $\dot{x}' = \omega(R_\oplus + h)\sin\theta$.

+ We are setting $\hat{z}'$ in the instantaneous $\hat{r}$ direction for simplicity. Corrections for apparent vertical cancel (I think)

must take into account b/c x' traveled is $\propto R_\oplus$

+ Subtlety: gravitational force is toward center of earth – we can't quite use uniform gravity. Using $R_\oplus \gg x', z'$, we have



$$m\ddot{z}' \approx -mg, \quad m\ddot{x}' \approx -mgx'/R_\oplus$$

+ We have $z' = h - \frac{1}{2}gt^2$, $x' = A\sin(\sqrt{g/R_\oplus}\,t)$ with $A = \omega(R_\oplus + h)\sin\theta\sqrt{R_\oplus/g}$

with $t \leq \sqrt{2h/g} \ll \sqrt{R_\oplus/g}$, $x' \approx A\left[\sqrt{g/R_\oplus}\,t - \frac{1}{6}(\sqrt{g/R_\oplus}\,t)^3 + \cdots\right]$

+

+ We now want *extra* x' distance object travels vs ground, which has x' speed $\omega R_\oplus \sin\theta$. We have

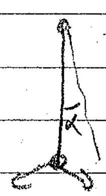
$$\Delta x' \approx \omega h \sin\theta\, t - \frac{1}{6}\omega R_\oplus \sin\theta\, (g/R_\oplus)\, t^3$$

When it hits the ground, $t = \sqrt{2h/g}$

$$\Delta x' \approx \frac{1}{3}\omega \sin\theta \sqrt{8h^3/g}$$

+ We have made basically the same approximation (small deviation from falling along z/z' with uniform gravity) and gotten same answer. Have to be more careful to account for everything here.

+ Notes on reading: KB makes *incorrect* statement about the inertial frame version. They say the object starts with same $v_{x'}$ as ground and speeds up as it falls due to angular momentum conservation. It actually starts faster b/c at greater height and slows down b/c z' velocity enters $\vec{J}$. Also, TM does inertial frame calculation using an elliptical orbit. Correct but needlessly complicated.



- Foucault Pendulum: This is a pendulum that can move in both $\hat{x}$ and $\hat{y}$ directions freely. Oscillates quickly along a line that rotates slowly in xy plane.
  + We take the small angle approximation, so gravity + tension act as restoring forces

$$F_z = -mg + T\cos\alpha, \quad F_{x,y} = -T\sin\alpha \begin{cases}\cos\phi \\ \sin\phi\end{cases}$$

At small angles,

$$\alpha \approx \sin\alpha \approx \frac{\sqrt{x^2+y^2}}{l}, \quad \cos\alpha \approx 1, \quad T \approx mg$$

b/c the pendulum moves on the surface of a sphere.

+ We also have $\vec{\omega} = \omega(\sin\theta\,\hat{y} + \cos\theta\,\hat{z})$, so the Coriolis force is

$$-2m\vec{\omega}\times\dot{\vec{r}} = +2m\omega\sin\theta\,\dot{x}\,\hat{z} - 2m\omega\cos\theta\,\dot{x}\,\hat{y} + 2m\omega\cos\theta\,\dot{y}\,\hat{x}$$

We can ignore Coriolis in $\hat{z}$ direction b/c it just changes T a small amount. (+ avgs to 0 over fast oscillation)

+ In total, we find equations of motion

$$\ddot{x} = -gx/l + 2\omega\cos\theta\,\dot{y} \quad , \quad \ddot{y} = -gy/l - 2\omega\cos\theta\,\dot{x}$$

To solve, set $u = x + iy$.

$$\ddot{u} + i2\omega\cos\theta\,\dot{u} + (g/l)\,u = 0$$

Define $\omega_0^2 = g/l$, $\omega_1 = \omega\cos\theta$.

+ Guess a solution $u = Ae^{i\Omega t}$. Then $-\Omega^2 - 2\omega_1\Omega + \omega_0^2 = 0$

$$\Rightarrow \Omega = -\omega_1 \pm \sqrt{\omega_1^2 + \omega_0^2} \approx -\omega_1 \pm \omega_0$$

That is

$$u = Ae^{-i\omega_1 t}e^{i\omega_0 t} + Be^{-i\omega_1 t}e^{-i\omega_0 t}$$

$$= e^{-i\omega_1 t}\left[Ae^{i\omega_0 t} + Be^{-i\omega_0 t}\right]$$

+ The factor in square brackets is set by init. cond. Suppose it is motion in x, so $A = B =$ real. The $e^{-i\omega_1 t}$ factor then slowly rotates it toward the y axis

+ Can understand from inertial frame if it is at the pole. The pendulum just oscillates in a line while the Earth rotates underneath.