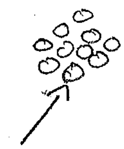
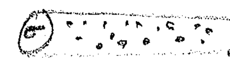
Scattering & Cross Sections

— The Mean Free Path

- Suppose a particle moves through a region of other particles that present cross-sectional area σ

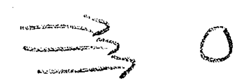


  + If you track the particle's path through the material, how many other particles does it hit? Reverse it: moving particle has cross-section σ, hits anything in cylinder around path



  + If the targets have number density n, # of collisions in length x is $= n\sigma x \Rightarrow$ average distance per collision is $\lambda = 1/n\sigma$. This is how far you expect a particle to get.
- Suppose there is a flux $f(x)$ of particles hitting the material. How does $f(x)$ change? Note: $f(x)$ = particle/area in beam/time.
  + Each collision removes a particle. But we just saw that the total collisions/time/area is $f(x)(n\sigma\, dx)$
  + Therefore, $f(x+dx) - f(x) = \frac{df}{dx}dx = -f\, n\sigma\, dx \Rightarrow \frac{df}{dx} = -f/\lambda$

    $\Rightarrow f(x) = f_0 e^{-x/\lambda}$.
- Turning this around, to the 1st picture again, consider a target and a flux f of incoming particles

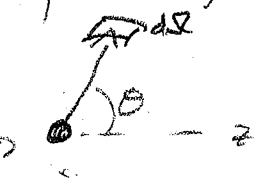


  + The cross-section is really a property of the scattering, but treat it as belonging to the target
  + We have defined the target cross-section σ the ratio of the rate of collisions to the incoming flux

    # collisions / time $= f\sigma$.

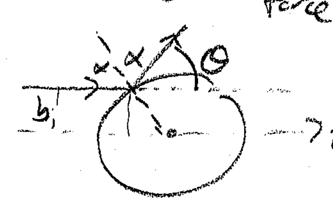
— Differential Cross Sections

- Cross Section is collisions / incoming flux. Want more specificity



  + If the initial velocity is along z axis w/ target at origin, the outgoing velocity is described by spherical polar angles θ, ϕ
  + We want # of scatterings with final velocity directed into a solid angle $d\Omega = \sin\theta\, d\theta\, d\phi$ at θ, ϕ
    Recall: solid angle = area of region on spherical surface / radius^2.
    4π solid angles on whole sphere, measured in <u>steradians</u>
  + This # of scatterings $\propto d\Omega$ but also $= f\, d\sigma$.
    We define the <u>differential cross section</u> $d\sigma/d\Omega$ as # scatterings per incoming flux per solid angle. The total cross section is $\sigma = \int d\Omega \left(\frac{d\sigma}{d\Omega}\right)$.

- <u>Ex</u> Hard-sphere Scattering
  + Suppose the target is a sphere of radius R. The total cross section is $\sigma = \pi R^2$ b/c the flux sees a circular profile (assuming contact force)
  + Let's determine the scattering angle θ for particle incoming w/ impact parameter b.



    For an elastic collision, the particle "reflects" around the radius. Therefore, $\theta = \pi - 2\alpha$, $\sin\alpha = b/R$.
    $\Rightarrow \quad b = R\cos(\theta/2)$
  + The flux that eventually scatters into a solid angle $d\Omega$ goes through area $d\sigma = |db|(b\, d\phi)$ where $db = -\frac{1}{2} R \sin(\theta/2)\, d\theta$
    (This is negative b/c larger impact parameters scatter less.)
  + Substituting, $d\sigma = \frac{1}{4} R^2 \sin\theta\, d\theta\, d\phi \Rightarrow d\sigma/d\Omega = R^2/4$
    This is isotropic and agrees with the expected total cross section.

- *Ex* Rutherford scattering
  - This is repulsive Coulomb potential scattering. Rutherford fired α particles (He nuclei) at gold foil. Electrons are too light to affect the α particle much, so $\frac{d\sigma}{d\Omega}(\theta)$ gives information on potential from nucleus. If nuclear charge fills the atom, potential is not Coulomb for small impact parameter
  - From our discussion of hyperbolic orbits, the impact parameter and scattering angle satisfy $b = \frac{k}{mv^2}\cot(\theta/2)$ with $k = \frac{q_1 q_2}{4\pi\epsilon_0}$ for the repulsive Coulomb potential
  - Then $d\sigma = b\,|db|\,d\phi$, and $db = \frac{-k}{2mv^2}\csc^2(\theta/2)\,d\theta$

    with a little algebra, $d\sigma/d\Omega = \frac{1}{4}\left(\frac{k}{mv^2}\right)^2 \frac{1}{\sin^4(\theta/2)}$.

    This is known as the Rutherford scattering cross section
  - The α particles can scatter to large angles, meaning there is a strong force at short distances $\Rightarrow$ nucleus is very small.
  - The total cross section is divergent. That's because the Coulomb force is infinite-range. There is always a tiny bit of scattering.