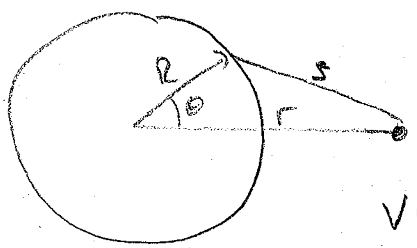
Inverse Square Law, Gravity, and Orbits

- Newton's Law of Universal Gravitation
  - The force of gravity between point objects of masses M and m is
    $\vec{F} = -\frac{GMm}{r^2}\hat{r}$, where $\vec{r}$ runs from M to m. Note: attractive
    + We will assume $M \gg m$, so M effectively doesn't move and can be located at the origin
    + Coulomb's Law for electrostatic force is similar, but can also be repulsive.
    + So we can write a general inverse-square force $\vec{F} = \frac{k}{r^2}\hat{r}$, k can be $\pm$.
  - The inverse square force is conservative.
    + The potential energy is therefore $V(\vec{r}) = k/r$ for point masses/charges
    + Why can we also use the inverse square law when one mass/charge is a large object? Consider gravity w/ uniform spherical shell M of surface density σ.



By the Law of cosines $s^2 = r^2 + R^2 - 2Rr\cos\theta$
Therefore

$$V(r) = -Gm \int \frac{\sigma R^2 \sin\theta \, d\theta \, d\phi}{\sqrt{r^2+R^2-2Rr\cos\theta}}$$

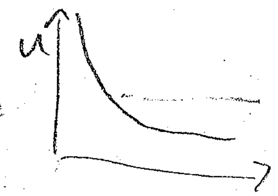
$$= -\frac{2\pi G m \sigma R^2}{2Rr}\left[(r+R) - |r-R|\right] = \begin{cases} -GMm/r & \text{for } r > R \\ -GMm/R & \text{for } r < R. \end{cases}$$

    + In other words, <u>outside</u> a spherical shell of mass (or charge), the potential energy is as for a point mass (charge).
      So therefore is the force!
    + Adding up shells lets you consider any spherically symmetric distributions. ~~This~~ follows easily from Gauss's law.
    + We will return to the question of potentials from non-point objects if time allows.

- Effective potential $U = \frac{k}{r} + \frac{J^2}{2mr^2}$ tells us type of motion

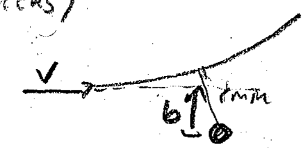
+ Repulsive Coulomb case $k > 0$

  At given E + J, there is a distance r_{min} of closest approach.



* Ex Say you have a heavy charge at the origin (nucleus)

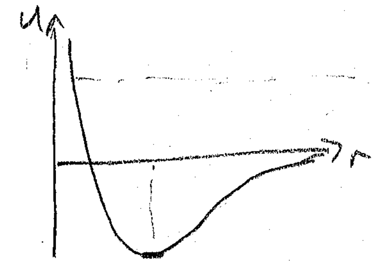
  A light charge of same sign approaches at impact parameter b and speed v. Then $J = mvb$, $E = \frac{1}{2}mv^2$



  The closest approach is given by $\dot{r} = 0$

$$\frac{1}{2}mv^2b^2\left(\frac{1}{r_{min}}\right)^2 + k\left(\frac{1}{r_{min}}\right) - \frac{1}{2}mv^2 = 0 \quad \text{(quadratic)}$$

+ The attractive case (gravity or opposite charges) $k < 0$ has several possibilities:



  $E > 0$ is similar to the repulsive case

  $E = 0$ is an object that barely escapes the attraction

  $E < 0$ is a closed orbit.

  $E = \min(U(r))$ must have $\dot{r} = 0 \Rightarrow$ circular orbit

+ Ex For a circular orbit, $\frac{dU}{dr} = \frac{|k|}{r^2} - \frac{J^2}{mr^3} = 0 \Rightarrow r = \frac{J^2}{m|k|}$

  This means the orbital speed is $v = v_\phi = r\dot{\phi} = r\left(\frac{J}{mr^2}\right) = \sqrt{|k|/mr}$.

  Near the surface of the earth, $GMm/r_\oplus^2 = mg$, so circular orbit speed is $v = \sqrt{r_\oplus g}$. An object moving that fast can go high enough that $\vec{F}_{grav} \neq$ constant. This means total energy is

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r_\oplus} = -\frac{1}{2}mr_\oplus g$$

  If this is launched vertically, it can reach $r = 2r_\oplus$

+ In general, for a circular orbit,

$$T = \frac{1}{2}mv^2 = \frac{|k|}{2r} = -\frac{1}{2}V.$$ This is an example of the <u>virial</u> theorem

- Orbit Solutions
  - Differential equation.
    + Remember $\dot\phi = J/mr^2$ always has the same sign, which we take >0 by choosing $\vec{J}$ along $\hat{k}$. So ϕ is monotonic in time $\Rightarrow$ can use it like a time coordinate.
    + It's also useful to change variables to $u = 1/r$ b/c of form of $U(r)$

    We have $$\dot r = \dot\phi \frac{dr}{d\phi} = -\frac{\dot\phi}{u^2}\frac{du}{d\phi} = -\frac{J}{m}\frac{du}{d\phi}$$

    + The total energy becomes
    $$E = \frac{J^2}{2m}\left[\left(\frac{du}{d\phi}\right)^2 + u^2\right] + ku$$

    If we complete squares
    $$\frac{J^2}{2m}u^2 + ku = \frac{J^2}{2m}\left(u + \frac{mk}{J^2}\right)^2 - \frac{mk^2}{2J^2}$$

    If we define $w = u + \frac{mk}{J^2}$
    $$\left(\frac{dw}{d\phi}\right)^2 + w^2 = \frac{2mE}{J^2} + \frac{m^2k^2}{J^4}$$

    + The RHS is constant, so differentiating gives
    $$2\frac{dw}{d\phi}\left[\frac{d^2w}{d\phi^2} + w\right] = 0 \Rightarrow [\;--\;] = 0$$

    The solution is $w = A\cos(\phi - \phi_0)$

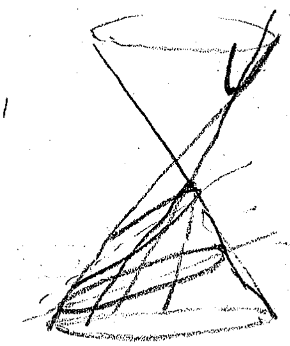
    + That is $\frac{1}{r} = -\frac{mk}{J^2} + A\cos(\phi-\phi_0)$. We note $\frac{J^2}{m|k|} \equiv l$ has units length.

    So define $A = e/l$, with e dimensionless.
    $$\frac{l}{r} = e\cos(\phi-\phi_0) \pm 1$$ w/ $\pm$ for attractive/repulsive potential.

    + Plugging back:
    $$e^2/l^2 = \frac{1}{l^2}\left(1 + \frac{2J^2E}{mk^2}\right)$$

    For a circular orbit, $E = -mk^2/2J^2 \Rightarrow e = 0$.

    For a closed orbit, $0 \le e < 1$. For an escaping orbit, $e \geq 1$.

- General properties of solutions
  - We will see that these are Conic sections, ie planar slices through a double cone

  - The origin $r=0$ is one focus
    e = eccentricity controls shape (E vs J)
    ℓ controls size = semi-latus rectum (J_{vs} V)
  - r is minimized when $\phi=\phi_0$. We might as well set $\phi_0=0$.
    This means $\phi=0$ is pericenter (perihelion around sun, perigee earth)
    For a closed orbit, furthest distance is apocenter (aphelion, apogee)
    The apsides are both apocenter + pericenter.
  - For attractive potentials, $r=\ell$ at $\phi=\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
- Elliptical (closed) orbits for $E<0$, $e<1$
  - Pericenter is at $r_{min}=x=\ell/(1+e)$, apocenter at $r_{max}=-x=\ell/(1-e)$
    $\Rightarrow$ The total displacement vs x is $r_{min}+r_{max}=\frac{2\ell}{1-e^2}\equiv 2a$.
    $\Rightarrow$ The center on the x axis is at $\frac{r_{min}-r_{max}}{2}=\frac{-\ell e}{1-e^2}=-ae$.
  - When x is at the center $r\cos\phi=-ae$, we have $y=r\sin\phi$
    $=\pm a\sqrt{1-e^2}\equiv\pm b$
  - So compare to ellipse of semimajor axis a, semiminor axis b, center at $x=-ae$. This is

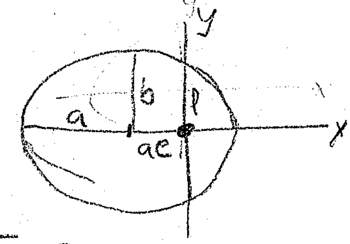
$$\frac{(x+ae)^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow (1-e^2)x^2+2ae(1-e^2)x+y^2=b^2-a^2e^2(1-e^2)$$
$$\Rightarrow x^2+y^2=e^2x^2-2e\ell x+\ell^2$$

    Meanwhile, the polar form is

$$(\ell-er\cos\phi)=r \Rightarrow x^2+y^2=\ell^2-2e\ell x+e^2x^2$$

  - Kepler's 1st Law of Planetary Motion: An orbit is an ellipse with the sun (or large object) at one focus. If both objects are similar size, the focus is the center of mass position

+ Relation of position to time follows from area swept out $\frac{dA}{dt} = \frac{J}{2m}$.



Since the total area of the ellipse is πab, period $T = 2\pi m ab/J$.

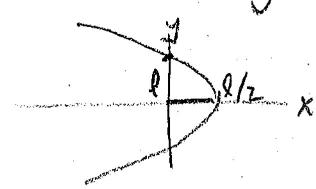
But note $b^2 = al$ and $J^2 = m|k|l = GMm^2 l \Rightarrow T^2 = 4\pi^2 a^3/GM$

+ $T^2 \propto a^3$ is Kepler's 3rd law. There is actually a small correction (next term) → for our solar system

+ In the solar system, distances measured in AU = astronomical units = semi-major axis of earth. Earth's eccentricity is $e = 0.0167$.

• Parabolic orbit $E = 0$, $e = 1$. Only for attractive potential.

+ The eqn of the orbit is $l = r\cos\phi + r \Rightarrow (l-x)^2 = x^2 + y^2$
$\Rightarrow x = \frac{l}{2} - \frac{y^2}{2l}$.



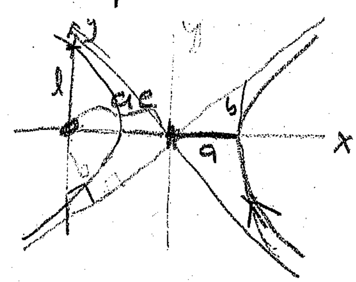
+ This is an object at escape velocity

• Hyperbolic orbit $E > 0$, $e > 1$. Valid for attractive or repulsive $V(r)$

+ The curve is $r(e\cos\phi \pm 1) = l$. For $\phi = 0$, $x_\pm = l/(e \pm 1)$. (2 different orbits)

$\Rightarrow$ Separation of orbits is $x_- - x_+ = \frac{2l}{e^2 - 1} \equiv 2a$

$\Rightarrow$ Center is at $x_+ + a = (e-1)a + a = ea$



+ The slope of the asymptotes is given by $\pm b/a$.

For very large x, y, the eqn is $ex \pm \sqrt{x^2 + (bx/a)^2} \simeq 0 \Rightarrow b = a\sqrt{e^2 - 1}$.

Note: b in figure 4.7 of KB is the impact parameter, fig B.2 is better.
It turns out the semi-axis b and the impact parameter are the same (below)

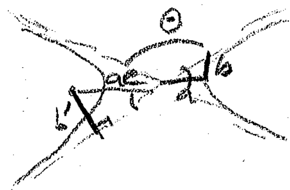
+ For comparison $l - ex = \pm r \Rightarrow x^2 + y^2 = l^2 - 2elx + e^2x^2$

while a hyperbola with focus at O, center at $x = ae$, semi-axes $a + b$ is

$$\frac{(x-ae)^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow x^2 + y^2 = l^2 - 2elx + e^2x^2 \text{ also.}$$

The close branch of the hyperbola is for attractive potentials; the far branch for repulsive.

+ Think of this as a scattering problem.



We note that $\sqrt{a^2+b^2} = ae$, so the 2 triangles shown are congruent by angle-side-angle theorem.

That means the impact parameter $b' = b$ semi-axis

+ Either hyperbolic orbit starts on 1 asymptote + switches to the other.

That means the object scatters by angle Θ.

We know $r \to \infty$ for $\phi = \pm \cos^{-1}(1/e) \Rightarrow \Theta = \pi - 2\cos^{-1}(1/e)$.

Geometrically, we also see $b/a = \sqrt{e^2-1} = \cot(\Theta/2)$.

With $\ell = J^2/m|k|$ and $e^2 = 1 + 2E\ell/|k|$, $b = \frac{\ell}{e^2-1} \cot \Theta/2 = \frac{J^2|k|}{2mE\ell} \cot \frac{\Theta}{2}$

$= \frac{|k|}{2E} \cot \Theta/2 = \frac{|k|}{mv^2} \cot \frac{\Theta}{2}$ where v = asymptotic speed.