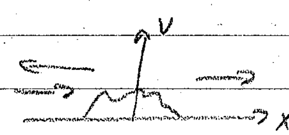
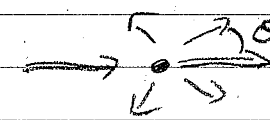
Scattering In 3D

Scattering + Cross Sections

- Comparison to 1D Scattering
  - In 1D, we generally consider a localized potential & build a non-normalizable energy eigenstate (scattering state). The stationary state has 3-parts outside the potential



    + An incident wave Ae^{ikx} moving right from the left.
    + A reflected wave Be^{-ikx} moving left on the left (superimposed on the incident wave)
    + A transmitted wave $Ce^{ik'x}$ moving right on the right. If the potential is at a different value on the right, $k' \neq k$.
    + The reflected wave is the part that scattered from the potential. The reflection coefficient $R = |B|^2$ tells you the amount of scattering
  - In 3D, there is a potential $V(\vec{r})$ localized near the origin (may or may not be spherically symmetric). What's the scattering state?

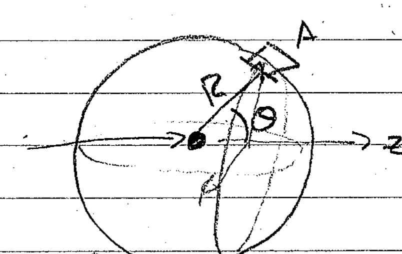


    + There is still an incident wave $\psi_i = e^{ikz}$ (at large r) which is also the transmitted wave. We've chosen it to be moving toward $+z$, but note it fills space.
    + It's a plane wave representing motion with a fixed momentum outside the potential
    + The scattered wave includes radial motion with different amplitudes in each direction
      $$\psi_s = f(\theta, \phi)\, e^{ikr}/r \quad \text{(at large } r\text{, outside potential)}$$
      We will see that the $1/r$ factor is necessary for the wavefunction to remain "normally" non-normalizable (Dirac normalizable)

+ A difference is that classically a particle does not always hit the potential. We want a more careful definition of "amount of scattering" than a reflection coeff.

- In both cases, the physical process is really time-dependent:
  + Lump of probability comes in, interacts with potential, scatters out. This is a lot like the classical picture of a particle coming in + bouncing out
  + We should really think of building the incoming lump as a wavepacket, or superposition of incoming waves. However, a lump with small momentum uncertainty has only a small range of k values for incoming waves, so the scattering state above can describe its behavior well.

— The Cross Section

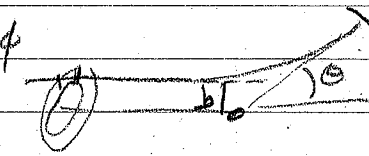


- Suppose we set up detectors in a spherical arrangement around the scattering potential.
  + How many particles scatter into the detector located at θ, ϕ?
  + θ = scattering angle, the amount that the particle changes direction, ϕ is the azimuthal angle around the original direction of motion.
  + By conservation of particle #, the number that enter the detector is proportional to A/R^2 where A = detector area and R = distance of detector from potential

- Solid Angle
  + The angle taken up by an arc of a circle is the arclength divided by the radius. Since the circumference is $2\pi R$, there are 2π radians in a circle.
  + The solid angle taken up by an area on a sphere is A/R^2. Therefore there are 4π steradians on a sphere.

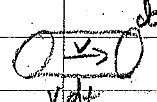
+ A "rectangular" area given by infinitesimal changes in θ, ϕ has solid angle $d\Omega \equiv \sin\theta \, d\theta \, d\phi$

- Differential Cross Section Classically
  + The number that enter the detector at θ, ϕ is the number that enter the region in such a way that they will scatter by angle θ $\rightsquigarrow$ This is deterministic classically.

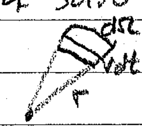


  + The impact parameter b is the closest distance the incident particle would get to the target if it did not deflect. Classically, b determines θ, and we often determine b as a function of θ.
  + Therefore, the # of particles that go into $d\theta \, d\phi$ at θ, ϕ is the number that pass through $db \, d\phi$ at impact parameter b.
  + This rate is the flux F of particles (number/area/time) entering the experiment times the cross sectional area $d\sigma = b \, db \, d\phi$. So the rate is $dN = F d\sigma$.
  + The ability of the potential to scatter should be independent of F. Since $dN \propto d\Omega$ of the detector, the property of the potential is the differential cross section
$$d\sigma/d\Omega \equiv \frac{1}{F}\frac{dN}{d\Omega}.$$
  + The (total) cross section is $\sigma = \int d\Omega \, d\sigma/d\Omega$. You can follow the example in the text to see that particles scattering from a hard sphere of radius R see cross section πR^2, as you'd expect.

- Quantum Differential Cross Section
  + Rather than number of particles (1) incident and (2) detected, we can ask about the amount of probability (1) incident + (2) detected
  + The wave number k of the incident wave is $k = p/\hbar$, so the wave has group velocity $v = \hbar k/m$. The probability passing area $d\sigma$ in time dt is $dP = |\psi_i|^2 d\sigma \, (v dt)$



+ Meanwhile, the probability scattered into a solid angle $d\Omega$ in time dt is



$$dP = |\psi_s|^2 (r^2 d\Omega)(v\,dt)$$

at the same group velocity

+ The differential cross section is therefore $\frac{d\sigma}{d\Omega} = r^2 \frac{|\psi_s|^2}{|\psi_i|^2} = |f(\theta,\phi)|^2$

The scattering amplitude $f(\theta,\phi)$ determines the cross section.

We want methods to calculate $f(\theta,\phi)$.

- Born Approximation: Solution for scattering amplitude $f(\theta,\phi)$ assuming it's small compared to the incident wave

— Integral form of Schrödinger Equation

• We're solving for a scattering state e'function, so the Schr. eqn is

$$\frac{-\hbar^2}{2m} \nabla^2 \psi + V\psi = E\psi \Rightarrow (\nabla^2 + k^2)\psi = (2m/\hbar^2) V\psi$$

+ For $V=0$, this is the Helmholtz eqn

This $(\nabla^2 + k^2)\psi = 0$ with $k = \sqrt{2mE}/\hbar$

+ With a nonzero potential, there is a source

$$(2m/\hbar^2)\, V(\vec{r})\, \psi(\vec{r})$$

• Green' functions

+ Suppose we have a diff. eqn. of the form

$$D^2 \psi(\vec{r}) = Q(\vec{r})$$

where D^2 is some 2nd order differential operator (in our case $D^2 = \nabla^2 + k^2$) and $Q(\vec{r})$ is a source

+ The Green's function $G(\vec{r})$ is defined as the solution to

$$D^2 G(\vec{r}) = \delta^3(\vec{r})$$

(Sometimes the δ-function is multiplied by a constant factor for convenience).

+ Then define

$$\psi(\vec{r}) = \int d^3\vec{r}\,'\, G(\vec{r}-\vec{r}\,')\, Q(\vec{r}\,')$$

Then

$$D^2\psi = \int d^3\vec{r}\,' \left[D^2 G(\vec{r}-\vec{r}\,')\right] Q(\vec{r}\,') = \int d^3\vec{r}\,'\, \delta^3(\vec{r}-\vec{r}\,')\, Q(\vec{r}\,') = Q(\vec{r})$$

+ The idea is that $G(\vec{r})$ is the response to a point source, and you can add them up for the solution with the full source.
+ You've done this many times: the electrostatic potential of a point charge is $q/4\pi\epsilon_0 r$, and the potential of a charge distribution is a sum over the potentials of individual point charges

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3\vec{r}\,' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

- Helmholtz equation Green's function
  + Text gives a derivation using (1) Fourier transform (2) residue theorem / Cauchy formula for the inverse transform (3) Choice of prescription for dealing with singularities. Last 2 parts involve complex analysis. If you've studied that + are interested in theoretical physics grad school, read it. I'll give you the sol'n.
  + The Green's function for the Helmholtz eqn is

$$G(\vec{r}) = -e^{ikr}/4\pi r$$

  + We can check that this works. Remember that $\nabla^2(1/r) = -4\pi\,\delta^3(\vec{r})$ cb/c electrostatic potential obeys the Poisson/Laplace eqn. Therefore,

$$\nabla^2 G = (-1/4\pi)\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\left(\frac{1}{r}e^{ikr}\right)\right)$$

$$\nabla^2 G = \frac{-1}{4\pi}\left\{e^{ikr}\nabla^2(1/r) - \frac{ik}{r^2}e^{ikr} + \frac{ik}{r^2}e^{ikr} - \frac{k^2}{r}e^{ikr}\right\}$$

$$= \delta^3(\vec{r})e^{ikr} + k^2 G. \text{ Since } e^{ik(0)} = 1, \text{ this checks.}$$

- Integral Schrödinger equation
  + The particular solution is therefore given by the integral over the Green's function. We can also add any solution to the $V=0$ (homogeneous) Schr. eqn.
  + So we have

$$\psi(\vec{r}) = \underbrace{\psi_0(\vec{r})}_{\text{homogeneous sol'n}} - \frac{m}{2\pi\hbar^2}\int d^3\vec{r}\,' \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}V(\vec{r}\,')\psi(\vec{r}\,')$$

  + This is a bit of a cheat: ψ is also under the integral. You can solve it iteratively or perturbatively (what we'll do)

- Born Series

- Long Distance Approx: We assume $V(\vec{r}')$ goes to zero outside a small region near the origin and want the wavefunction far away.
  + Then when $|\vec{r}|$ is large, we can write
    $$|\vec{r}-\vec{r}'| = \sqrt{r^2+r'^2-2\vec{r}\cdot\vec{r}'} \simeq r - \vec{r}\cdot\vec{r}'/r + \cdots$$
  + The Green's function is approximately
    $$G(\vec{r}-\vec{r}') \simeq \frac{1}{r} e^{ikr} e^{-ik\hat{r}\cdot\vec{r}'}$$
  + Note that we can take the simplest approximation for the denominator but need the next for the exponential, which can change significantly if $k|\vec{r}'| \simeq 1$
  + The wavefunction at large r becomes (with $\psi_0(\vec{r}) = e^{ikz}$)
    $$\psi(\vec{r}) = e^{ikz} - \frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int d^3\vec{r}'\, e^{-ik\hat{r}\cdot\vec{r}'} V(\vec{r}')\psi(\vec{r}')$$
    so
    $$f(\theta,\phi) = \frac{-m}{2\pi\hbar^2}\int d^3\vec{r}'\, e^{-ik\hat{r}\cdot\vec{r}'} V(\vec{r}')\psi(\vec{r}')$$

- Born Approximation (Max Born).
  + The difficulty with our formula for the scattering amplitude is that it's recursive: it depends on the wavefunction we're trying to find.
  + If we assume $V(\vec{r}')$ is small (compared to the state's energy $\hbar^2k^2/2m$), $\psi(\vec{r}) \simeq e^{ikz} + \delta\psi$, where $\delta\psi$ is small
  + But then $V\delta\psi$ in the integral is doubly small, so we can ignore it
  + The 1st order Born approximation is
    $$f(\theta,\phi) = \frac{-m}{2\pi\hbar^2}\int d^3\vec{r}'\, e^{ik(\hat{z}-\hat{r})\cdot\vec{r}'} V(\vec{r}')$$
  + You can go to higher orders in V by substituting back repeatedly (~~use~~ $\alpha = m/2\pi\hbar^2$)
    $$\psi = \psi_0 + \alpha\int d^3\vec{r}_1\, G V\psi_0 + \alpha^2\int d^3\vec{r}_1\int d^3\vec{r}_2\, GVGV\psi_0 + \cdots$$
    This is the Born series.

- Low Energies
  + If $k|\vec{r}'| \ll 1$ everywhere V is non-negligible, we can drop the complex exponential. This is when the wavelength $\lambda = 2\pi/k \gg$ the size of the region containing V, which is also when the kinetic energy is small compared to $\hbar^2/2m(\text{size})^2$ (but still bigger than V).
  + We have $f(\theta,\phi) = \frac{-m}{2\pi\hbar^2}\int d^3\vec{r}\ V(\vec{r})$
  + Example: "Soft-sphere Scattering"

$$V(\vec{r}) = V_0 \text{ for } r<a, \quad = 0 \text{ for } r>a.$$

Then $f(\theta,\phi) = -(m/2\pi\hbar^2)\, V_0\, (4\pi a^3/3)$

and $d\sigma/d\Omega = |f|^2 = \left(2ma^3V_0/3\hbar^2\right)^2, \quad \sigma = 4\pi\left(2ma^3V_0/3\hbar^2\right)^2$

  + Note that you can make $V(\vec{r})$ non-spherically symmetric, but $d\sigma/d\Omega$ = constant always in this approx.

- Spherical Symmetry
  + Define $\vec{\kappa} = k(\hat{z}-\hat{r})$. Then $-\hbar\vec{\kappa}$ = momentum transfer of the scattering. Note $\kappa^2 = k^2(2-2\hat{z}\cdot\hat{r}) = 2k^2(1-\cos\theta) = 4k^2\sin^2(\theta/2)$
  + To do the integral for $f(\theta,\phi)$, choose the z'-axis (ie, for $\vec{r}'$) to lie along κ. Then

$$f(\theta,\phi) = \frac{-m}{2\pi\hbar^2}\int d^3\vec{r}'\ e^{i\kappa r'\cos\theta'}\ V(\vec{r}')$$

  + If $V(\vec{r}') = V(r')$ is spherically symmetric, the angular integral is

$$2\pi\int_0^\pi d\theta' \sin\theta'\, e^{i\kappa r'\cos\theta'} = 2\pi\int_{-1}^{1} du\ e^{i\kappa r' u} = \frac{4\pi}{\kappa r'}\sin(\kappa r')$$

Therefore,

$$f(\theta,\phi) = -\frac{2m}{\hbar^2\kappa}\int_0^\infty dr\ rV(r)\sin(\kappa r)$$

  + With spherical symmetry, $f(\theta,\phi)$ is independent of ϕ, but it can depend on θ b/c the incident wave picks out a direction.

+ Example: Yukawa Potential (Screened Coulomb Potential)

This is $V = \beta e^{-\mu r}/r$ (so it's important for $r \lesssim 1/\mu$)

We have

$$f(\theta) = -2m\beta/\hbar^2\kappa \int_0^\infty dr\, e^{-\mu r} \sin(\kappa r)$$

$$= \frac{-2m\beta}{\hbar^2\kappa}\left(\frac{1}{2i}\right)\left[-\frac{1}{i\kappa-\mu} + \frac{1}{-i\kappa-\mu}\right] = \frac{-2m\beta}{\hbar^2\kappa}\,\frac{1}{2i}\,\frac{2i\kappa}{\mu^2+\kappa^2} = \frac{-2m\beta}{\hbar^2(\mu^2+\kappa^2)}$$

The Rutherford scattering cross section is for the Coulomb potential or $\mu \to 0$. Then

$$f(\theta) = -\left(2m/\hbar^2\kappa^2\right)\left(q_1q_2/4\pi\epsilon_0\right) = \frac{-q_1q_2}{4\pi\epsilon_0}\,\frac{1}{4E\sin^2(\theta/2)}$$

The differential cross section is

$$d\sigma/d\Omega = \left(\frac{q_1q_2}{4\pi\epsilon_0}\right)^2 \frac{1}{16E^2\sin^4(\theta/2)}$$ same as classical!