PHYS-4602 Homework 7 Due 15 March 2021

This homework is due to `https://uwcloud.uwinnipeg.ca/s/ptx3smosp2xFtmE` by 10:59PM on the due date. You may submit a PDF either scanned from handwriting or generated with LaTeX or a word processor (with an equation editor).

1. **Stark Effect** *based on Griffiths 6.36*

   The presence of an external electric field $E_0\hat{z}$ shifts the energy levels of a hydrogen atom, which is called the Stark effect. Consider the hydrogen atom to be described by the Coulomb potential; the external electric field introduces a perturbation

   $$H_1 = eE_0 z = eE_0 r\cos\theta \ . \tag{1}$$

   We have already seen on homework that the expectation value of this Hamiltonian in the ground state $n = 1$ vanishes, so there is no shift in the ground state energy. In this problem, we consider the degenerate perturbation theory of the $n = 2$ states. *As spin does not enter, do not consider it in this problem.*

   (a) The four states $|2,0,0\rangle$, $|2,1,0\rangle$, and $|2,1,\pm1\rangle$ are degenerate at 0th order. Label these states sequentially as $i = 1,2,3,4$. Show that the matrix elements $W_{ij} = \langle i|H_1|j\rangle$ form the matrix

   $$W = -3aeE_0 \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \ , \tag{2}$$

   where empty elements are zero and a is the Bohr radius. *Hint*: Note that L_z commutes with H_1, so only states with the same quantum number m can have nonzero matrix elements; this will save you quite a bit of work. Then use the angular wavefunctions to see that all the diagonal elements of W must vanish. Finally, use the explicit wavefunctions to evaluate the remaining matrix elements of W (there should only be one independent one left).

   (b) Diagonalize this matrix to show that $|\pm\rangle = (1/\sqrt{2})(|2,0,0\rangle \pm |2,1,0\rangle)$ are eigenstates of W. Find the first order shift in energies of $|\pm\rangle$. *Hint*: Note that the corrected eigenstates may still have contributions from other values of the principal quantum numbers n, but that doesn't matter for this purpose.

2. **Matrix Perturbation Theory**

   Consider the matrix Hamiltonian

   $$H \simeq \begin{bmatrix} E_1 & \epsilon \\ \epsilon & E_2 \end{bmatrix} \tag{3}$$

   with $E_1 \neq E_2$ except when you are told otherwise. Assume that $\epsilon \ll E_1, E_2$.

   (a) To first order in perturbation theory, find the energy eigenvalues and eigenstates.

   (b) What is the first order correction to the energy if $E_1 = E_2 = E$?

   (c) Find the energy eigenvalues to second order in perturbation theory.

   (d) Find the energy eigenvalues and eigenstates exactly. Then expand them as a power series in ϵ and compare to your perturbative answers from parts (a,c). In the case that $E_1 = E_2 = E$, how does your answer compare to part (b)?

3. **Localized Magnetic Field**

Two electrons are localized at well-separated lattice sites, so they can be treated as distinguishable particles. The two electrons interact with each other, and only the first electron experiences a magnetic field. The Hamiltonian is $H = A\vec{S}_1 \cdot \vec{S}_2 + BS_{1,z}$, where $\vec{S}_j$ is the spin of electron j and A, B are constants. (The B term represents the magnetic field on the first electron.)

(a) Assume that there is no magnetic field, so $B = 0$. Find the energy eigenstates and eigenvalues. Which eigenstate is the ground state assuming $A > 0$?

(b) Assume that the magnetic field on electron 1 is small ($B \ll \hbar A$) and find the ground state and ground state energy to first order in B. *Hint:* make sure you can write your eigenstates from the previous part in terms of the individual spins.