PHYS-4602 Homework 5 Due 22 Feb 2021

This homework is due to `https://uwcloud.uwinnipeg.ca/s/ptx3smosp2xFtmE` by 10:59PM on the due date. You may submit a PDF either scanned from handwriting or generated with LaTeX or a word processor (with an equation editor).

1. **Cloning Means FTL Communication** *based on a problem by Wilde*

   Suppose that Alice and Bob are at two ends of an EPR/Bell experiment. In other words, they are at rest with respect to each other and separated by 5 lightyears, and each receives one of a pair of entangled electrons with total spin state $s = 0$ simultaneously (in their common rest frame). By prior agreement, Alice measures either the S_z or S_x spin of her electron as soon as she receives it, but Bob does not know which spin she measures.

   After Alice's measurement (in their rest frame time), Bob's electron is in some state $|\psi\rangle_B$. Suppose, in contradiction to the no-cloning theorem, Bob can clone his electron's state onto a large number N of other electrons. (For example, Bob can do some quantum operation that takes his $N+1$ electrons from state $|\psi\rangle_B|\uparrow\rangle_1 \cdots |\uparrow\rangle_N$ to state $|\psi\rangle_B|\psi\rangle_1 \cdots |\psi\rangle_N$.) What measurement(s) can Bob do on his extra N electrons that will tell him with great certainty whether Alice measured the S_z or S_x spin of her electron? Explain your answer. (Note that Bob can accomplish his measurement before Alice can tell him her measurement choice, so they can establish faster-than-light communication in this way. This is a good reason for the no-cloning theorem!)

2. **Coherent Teleportation**

   Consider a system of 3 qubits. We will demonstrate the existence of *coherent teleportation*, which transfers a state $|\psi\rangle$ from qubit 1 to qubit 3 without requiring measurement.

   For notation, a subscript i on a 1-qubit operator means it acts on qubit i, so $\mathbb{H}_i$ is the Hadamard operator acting on qubit i. We also define $\Gamma_{i,j}$ as the CNOT operator acting on qubit j with qubit i as control. For example, $\Gamma_{1,2}$ is the usual CNOT operator that reverses qubit 2 if qubit 1 is $|1\rangle$, while $\Gamma_{3,1}$ reverses qubit 1 if qubit 3 is $|1\rangle$, etc. Finally, we denote $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ for any qubit.

   (a) Show that
   $$\frac{1}{\sqrt{2}}\left(|+\rangle_1|+\rangle_2 + |-\rangle_1|-\rangle_2\right) = \frac{1}{\sqrt{2}}\left(|0\rangle_1|0\rangle_2 + |1\rangle_1|1\rangle_2\right) \ . \tag{1}$$

   (b) We wish to define a new operator $\Delta_{i,j}$ that takes $\Delta_{i,j}|+\rangle_i|0\rangle_j = |+\rangle_i|+\rangle_j$ and $\Delta_{i,j}|-\rangle_i|0\rangle_j = |-\rangle_i|-\rangle_j$. Show that $\Delta_{i,j} \equiv \mathbb{H}_i\mathbb{H}_j\Gamma_{i,j}\mathbb{H}_i$ obeys these equations. With this definition, find $\Delta_{i,j}|+\rangle_i|1\rangle_j$ and $\Delta_{i,j}|-\rangle_i|1\rangle_j$.

   (c) Show that $\Delta_{i,j}$ is not a cloning operator. *Hint:* If $|\psi\rangle = a|+\rangle + b|-\rangle$, show that $\Delta_{1,2}|\psi\rangle_1|0\rangle_2 \neq |\psi\rangle_1|\psi\rangle_2$ by comparing both sides.

   (d) Let $|\psi\rangle = a|0\rangle + b|1\rangle$ be a general qubit. Show that
   $$\Gamma_{3,2}\Delta_{1,2}\Gamma_{1,3}|\psi\rangle_1|0\rangle_2|0\rangle_3 = \frac{1}{\sqrt{2}}\Big(|0\rangle_1|0\rangle_2 + |1\rangle_1|1\rangle_2\Big)|\psi\rangle_3 \ . \tag{2}$$

3. **Quantum Reality or Not**

To answer this question, you will need to watch the video of Sidney Coleman's famous lecture "Quantum Mechanics In Your Face" at `http://media.physics.harvard.edu/video/?id=SidneyColeman_QMIYF` or `https://www.youtube.com/watch?v=EtyNMlXN-sw` . (This is about an hour and supplements the reading, which is not long this week; the transcript is at `https://arxiv.org/pdf/2011.12671.pdf`.)

(a) The Bell experiment considers 2 distinguishable spin 1/2 particles in the singlet ($s = 0$) total spin state. If $\hat{a}$ and $\hat{b}$ are two unit vectors, show that

$$\left\langle \left(\hat{a} \cdot \vec{S}^{(1)}\right) \left(\hat{b} \cdot \vec{S}^{(2)}\right) \right\rangle = -\frac{\hbar^2}{4} \hat{a} \cdot \hat{b} \,. \tag{3}$$

Hint: Think about a convenient choice of axes and remember that the spin operators are given in matrix form as $S_i \simeq (\hbar/2)\sigma_i$ in terms of the Pauli matrices.

(b) Three electrons are prepared in the so-called "GHZM" spin state

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle_1 |\uparrow\rangle_2 |\uparrow\rangle_3 - |\downarrow\rangle_1 |\downarrow\rangle_2 |\downarrow\rangle_3) \tag{4}$$

described in the video. Show that $|\psi\rangle$ is an eigenstate of the operator $S_x^{(1)} S_y^{(2)} S_y^{(3)}$ and find the eigenvalue.