PHYS-4602 Homework 4 Due 8 Feb 2021

This homework is due to `https://uwcloud.uwinnipeg.ca/s/ptx3smosp2xFtmE` by 10:59PM on the due date. You may submit a PDF either scanned from handwriting or generated with LaTeX or a word processor (with an equation editor).

1. **1-Qubit von Neumann Entropy**

   On the previous assignment, we saw that the general density matrix of a single qubit is

   $$\rho = \frac{1}{2}\left(1 + \vec{a}\cdot\sigma\right) = \frac{1}{2}\begin{bmatrix} 1+a_3 & a_1 - ia_2 \\ a_1 + ia_2 & 1-a_3 \end{bmatrix} , \tag{1}$$

   where $\vec{a} = (a_1, a_2, a_3)$ is a vector describing the mixed state for $\vec{a}^2 < 1$ (and σ are the Pauli matrices). Find the von Neumann entropy of this general density matrix. *Hint:* start by finding the eigenvalues.

2. **Entropy Inequalities and Geometry**

   Recent developments in quantum gravity have tried to link quantities from quantum information theory, like von Neumann entropy, to geometry. In this question we will consider a toy model for the entropy in terms of geometry and see if it is allowed by the properties of the entropy.

   To start, imagine a system with various pieces 1,2,$\cdots$. Associate the reduced density matrix ρ_i of each part with a vector $\vec{x}_i$ and the entropy with the vector length $S(\rho_i) = |\vec{x}_i|$. Finally, the density matrix of two parts of the system put together is associated with the sum of the two corresponding vectors: $\rho_{i,j} \to \vec{x}_i + \vec{x}_j$, so $S(\rho_{i,j}) = |\vec{x}_i + \vec{x}_j|$. We do not put any restrictions on the vectors $\vec{x}_i$.

   (a) Show that this formula for the entropy satisfies both the Araki-Lieb inequality and the subadditivity property for a system with two parts.

   (b) On the other hand, suppose the system has three parts and find a counterexample to show that our prescription for the entropy does not satisfy strong subadditivity, so it cannot be valid (unless additional restrictions are imposed on the vectors $\vec{x}_i$).

3. **2-Qubit Gates**

   Consider a 2 qubit system. Choose a basis for the 2 qubit Hilbert space and use it for all parts of this problem.

   (a) Write the CNOT gate operator as a matrix in that basis and show that it is unitary.

   (b) Consider the 1 qubit gate NOT acting only on the first qubit of our two. Write this gate (call it NOT_1) as a matrix in your 2-qubit basis.

4. **Square Root of NOT**

   In quantum computing, the NOT gate reverses $|0\rangle$ and $|1\rangle$ qbits. In the standard matrix form, NOT is represented by the Pauli matrix σ_x.

   (a) Show that the operator U given by the matrix

   $$U \simeq \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} \tag{2}$$

is the “square root” of NOT in the sense that $U^2 \simeq \sigma_x$ and also demonstrate that U is unitary ($U^\dagger U = 1$).

(b) Consider the matrix $e^{-i\theta} \exp(i\theta\sigma_x)$. Show that this is U for $\theta = \pi/4$ and NOT for $\theta = \pi/2$, also demonstrating that U is the square root of NOT.

(c) Find the operator $e^{i\pi/4}U$ in terms of the phase rotation gate $R(\phi)$ and the Hadamard gate $\mathbb{H}$. You will need to choose a particular value for the phase ϕ.