PHYS-4602 Homework 3 Due 1 Feb 2020

This homework is due to `https://uwcloud.uwinnipeg.ca/s/ptx3smosp2xFtmE` by 10:59PM on the due date. You may submit a PDF either scanned from handwriting or generated with LaTeX or a word processor (with an equation editor).

1. **1-Qbit Density Matrix** *inspiration from Griffiths & Schroeter 12.6 & 12.8*

   Consider the density matrix ρ for a single qbit (you may consider this to be the spin of a single spin-1/2 particle instead). In this problem, describe ρ as a matrix rather than an abstract operator.

   (a) Prove that $\rho^2 = \rho$ if and only if the state is pure. *Hint:* Think about the diagonal form of ρ in pure and mixed states.

   (b) Using the requirements that $\mathrm{tr}(\rho) = 1$ and $\rho^\dagger = \rho$, show that the most general density matrix for a single qbit is

   $$\rho = \frac{1}{2}\begin{bmatrix} (1+a_3) & (a_1 - ia_2) \\ (a_1 + ia_2) & (1-a_3) \end{bmatrix} , \tag{1}$$

   where $a_{1,2,3}$ are real numbers. (This can also be written in terms of the Pauli sigma matrices as $(1 + \vec{a}\cdot\vec{\sigma})/2$.)

   (c) Define the *Bloch vector* $\vec{a}$ as the vector with components $a_{1,2,3}$. Use part (a) to show that ρ represents a pure state if $|\vec{a}| = 1$ and a mixed state if $|\vec{a}| < 1$ (that is, the Bloch vector lies on the surface of the Bloch sphere for a pure state and inside the Bloch sphere for a mixed state).

   (d) If the state is a pure state $|\psi\rangle = \cos\theta|0\rangle + e^{i\phi}\sin\theta|1\rangle$, show that $\vec{a}$ is a unit vector with polar angle 2θ and azimuthal angle ϕ (that is, $\vec{a} = [\cos\phi\sin(2\theta), \sin\phi\sin(2\theta), \cos(2\theta)]^T$).

2. **Traces of Operators**

   Use Dirac bra/ket notation and the completeness relation for an orthonormal basis to prove the following (you may assume that the Hilbert space is finite-dimensional):

   (a) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for any two operators A, B

   (b) The expectation value $\langle A\rangle = \mathrm{tr}(A\rho)$ for any observable A in a mixed state with density operator ρ

3. **Total Spin and Mixed States**

   Consider a system composed of two spins of $s = 1$ combined into a total spin 2 state

   $$|s=2, m=0\rangle = \frac{1}{\sqrt{6}}|1,1\rangle_1|1,-1\rangle_2 + \sqrt{\frac{2}{3}}|1,0\rangle_1|1,0\rangle_2 + \frac{1}{\sqrt{6}}|1,-1\rangle_1|1,1\rangle_2 \ . \tag{2}$$

   Write the reduced density operator for the first spin after the partial trace of the second spin. Is this a mixed state? Is it maximally mixed?

4. **Thermal State**

   The density matrix for a quantum system in thermal equilibrium at temperature T is

   $$\rho = e^{-H/k_BT}/Z \text{ where } Z \equiv \mathrm{tr}\left(e^{-H/k_BT}\right) \tag{3}$$

   is the *partition function*, H is the Hamiltonian, and k_B is the Boltzmann constant.

(a) First, show that the partition function is

$$Z = \sum_n e^{-E_n/k_B T} \ , \tag{4}$$

where the E_n are the energy eigenvalues.

(b) Show that another expression for the density operator is

$$\rho = \frac{1}{Z} \sum_n e^{-E_n/k_B T} |E_n\rangle\langle E_n| \ , \tag{5}$$

where $|E_n\rangle$ are the energy eigenvectors.

(c) Show that the so-called *thermofield double state*

$$|\psi\rangle = \frac{1}{\sqrt{Z}} \sum_n e^{-E_n/2k_B T} |E_n\rangle_1 |E_n\rangle_2 \ , \tag{6}$$

where $1, 2$ indicate two copies of the quantum system, is a purification of the thermal state by showing that the partial trace of $|\psi\rangle\langle\psi|$ over system 2 is ρ.

5. **3-Particle States** *from some Griffiths problems*

Consider three particles, each of which is in one of the single-particle states $|\alpha\rangle$, $|\beta\rangle$, or $|\gamma\rangle$, which are orthonormal.

(a) If the particles are bosons, write down the state where one particle is in each of $|\alpha\rangle$, $|\beta\rangle$, and $|\gamma\rangle$. *Hint*: This state must be symmetric under the exchange of *any* pair of the bosons.

(b) Write down all possible 3-particle states (including normalization) with two particles in the same 1-particle state and the third particle in a different 1-particle state, still in the case that the particles are indistinguishable bosons.

(c) How many linearly independent states can you form if the particles are fermions? Write down all the possible linearly independent states. *Hint*: Similarly to the above, these states must be antisymmetric under the exhange of *any* pair of the fermions.