PHYS-4602 Homework 1 Due 18 Jan 2020

This homework is due to `https://uwcloud.uwinnipeg.ca/s/ptx3smosp2xFtmE` by 10:59PM on the due date. You may submit a PDF either scanned from handwriting or generated with LaTeX or a word processor (with an equation editor).

1. **Dual Vectors and Change of Basis**

   Consider a 3-dimensional complex column vector space, which has the usual orthonormal basis

$$|e_1\rangle \simeq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \ , \ |e_2\rangle \simeq \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \ , \ |e_3\rangle \simeq \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \ . \tag{1}$$

   In that basis, the vectors $|f_i\rangle$ $(i = 1, 2, 3)$ can be written as

$$|f_1\rangle \simeq \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \ , \ |f_2\rangle \simeq \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \ , \ |f_3\rangle \simeq \frac{1}{\sqrt{6}} \begin{bmatrix} i \\ -i \\ -2i \end{bmatrix} \ . \tag{2}$$

   (a) Write the $|f_i\rangle$ as linear superpositions of the $|e_i\rangle$ basis vectors.

   (b) Show that the $|f_i\rangle$ are normalized and mutually orthogonal, so they form a complete orthonormal basis (distinct from the set of $|e_i\rangle$).

   (c) Write the associated dual vectors $\langle f_i|$ as row vectors in the $\{\langle e_i|\}$ basis.

2. **Diagonalization** *Based on Griffiths A.26*

   Consider a three-dimensional Hilbert space with orthonormal basis $|e_i\rangle$, $i = 1, 2, 3$. The operator A takes the matrix representation

$$A = \sum_{i,j} |e_i\rangle\langle e_i|A|e_j\rangle\langle e_j| \simeq \begin{bmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{bmatrix} \ . \tag{3}$$

   You should be able to check yourself that A is Hermitian.

   (a) Find the eigenvalues a_i and corresponding eigenstates $|a_i\rangle$ $(A|a_i\rangle = a_i|a_i\rangle)$ written in terms of their components $\langle e_j|a_i\rangle$. Choose the eigenstates to form an orthonormal eigenbasis; that is, choose any ambiguities such that $\langle a_i|a_j\rangle = \delta_{ij}$.

   (b) As stated in class, A can be written in the form

$$A = \sum_i a_i |a_i\rangle\langle a_i| \ , \tag{4}$$

   where a_i are the eigenvalues and $|a_i\rangle$ are the eigenvectors of A. Verify that formula (4) gives the same operator as (3) when you plug in your answer to part (a) for the eigenvalues and eigenvectors.

   (c) Write the state $|\psi\rangle = |e_1\rangle - i|e_3\rangle$ in the A eigenbasis (as a superposition of the $|a_i\rangle$).

3. **Permutation Operator**

   Consider an N-dimensional Hilbert space with orthonormal basis $\{|1\rangle, |2\rangle, \cdots |N\rangle\}$ and define the permutation operator S such that $S|n\rangle = |n+1\rangle$ for $1 \leq n < N$ and $S|N\rangle = |1\rangle$. (S is the same as translation by one site on a lattice of allowed positions.)

(a) Show that the state

$$|\lambda\rangle = \sum_{n=1}^{N} \lambda^{-n+1}|n\rangle \tag{5}$$

is an eigenstate of S with eigenvalue λ as long as λ takes one of N allowed values. Find those allowed values.

(b) Is S ever a Hermitian operator? If so, what are the values of N such that S is Hermitian?

(c) In the orthonormal basis described, write S in matrix form for the cases of $N = 2$ and $N = 3$.

4. **Commutators and Functions of Operators**

(a) Suppose $|a\rangle$ is an eigenfunction of some operator A, $A|a\rangle = a|a\rangle$. Consider the inverse operator A^{-1} defined such that $AA^{-1} = A^{-1}A = 1$. Show that $|a\rangle$ is an eigenvector of A^{-1} with eigenvalue $1/a$ if $a \neq 0$ (if there is an eigenvalue $= 0$, A is not invertible).

(b) For any function $f(x)$ that can be written as a power series

$$f(x) = \sum_n f_n x^n \ , \tag{6}$$

we can define

$$f(A) = \sum_n f_n A^n \ , \tag{7}$$

where A^n denotes operating with A n times. Show that

$$f(A)|a\rangle = f(a)|a\rangle \ . \tag{8}$$

Does this result hold if the power series includes negative powers?

(c) For any three operators A, B, C, show that

$$[A, BC] = [A, B]C + B[A, C] \ . \tag{9}$$

(d) Then prove by induction that

$$[A, B^n] = n[A, B]B^{n-1} \ , \tag{10}$$

if $[A, B]$ commutes with B (for $n > 0$).