PHYS-3203 Homework 7 Due 10 March 2021

This homework is due to `https://uwcloud.uwinnipeg.ca/s/T6ykcP988pa3kpG` by 10:59PM on the due date. You may submit a PDF either scanned from handwriting or generated with LaTeX or a word processor (with an equation editor).

The first two questions are questions from last year's midterm test. Together, they were worth about 45% of the test. A version of of the first problem from assignment #4 was also on that midterm.

1. **Former Midterm Question #1**

   A bead of mass $2m$ and two beads of mass m slide on a circular hoop of radius R with no friction. Springs of spring constant k stretch from the heavy bead to each of the lighter beads, and a spring of constant $\bar{k}$ stretches between the two lighter beads. Write down a matrix equation in the form of an eigenvalue/eigenvector problem that allows you to find the normal mode frequencies but do **not** attempt to solve for the frequencies.

2. **Former Midterm Question #2**

   The support point of a symmetric top with mass M is free to slide along a frictionless surface. The top's moments of inertia about its center of mass are $I_1 = I_2 = I$ and I_3, and its center of mass is a distance R from the support. The Lagrangian for this system is

   $$L = \frac{1}{2}M\left(\dot{x}^2 + \dot{y}^2 + R^2\sin^2(\theta)\dot{\theta}^2\right) + \frac{1}{2}I\left(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2\right) + \frac{1}{2}I_3\left(\dot{\psi} + \cos(\theta)\dot{\phi}\right)^2 - MgR\cos\theta \quad (1)$$

   in terms of the center of mass coordinates $x, y, z = R\cos\theta$ and Euler angles ψ, θ, ϕ.

   (a) (20 points) Find the Hamiltonian of this system.

   (b) List four conserved quantities for this system and explain very briefly how you know they are conserved.

3. **Hanging Springs** *based on a Kibble & Berkshire problem*

   Two identical springs of spring constant k are both attached to the same object of mass m. The other end of the first spring is attached to a fixed support, while the other end of the second spring is attached to an object of the same mass m. When they are held horizontal, each spring has equilibrium length l. Consider instead orienting the springs so that the first hangs from the ceiling and the second hangs downward from the object in the middle. Define generalized coordinates for the masses such that $l + x_1$ is the length of the first spring and $l + x_2$ is the length of the second.

   (a) Write the potential energy of the system in terms of x_1, x_2. Then find the equilibrium positions x_1^0, x_2^0 of the masses by minimizing the potential.

   (b) Now define generalized coordinates $y_{1,2}$ that are the displacements from the vertical equilibrium, ie, $x_{1,2} = x_{1,2}^0 + y_{1,2}$. Find the Lagrangian in terms of the $y_{1,2}$ coordinates.

   (c) Find normal modes and the frequencies of oscillation. Describe the motion for each normal mode in terms of the ratio y_2/y_1. Use the generalized eigenvalue problem from the equations of motion.

4. **Zero Mode** *From Thornton & Marion*

Three coupled oscillators have the same mass m and potential energy

$$V = \frac{k_1}{2}(x_1^2 + x_3^2) + \frac{k_2}{2}x_2^2 + \sqrt{\frac{k_1 k_2}{2}}(x_1 x_2 + x_2 x_3)\ . \tag{2}$$

(a) Find the normal mode frequencies. *Hint:* One should be zero.

(b) Find the (unnormalized) normal mode for the zero frequency.

(c) Solve the equation of motion for the zero frequency normal coordinate. What does this represent? What is the potential energy of this normal coordinate? *Hint:* Given the frequency, we know the normal coordinate EOM; you do not need to find the normal coordinate in terms of the x coordinates.