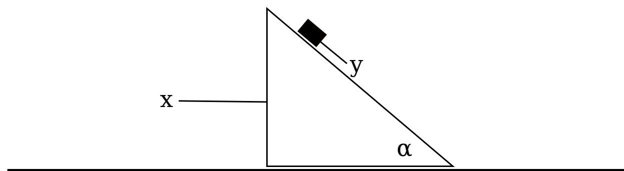
PHYS-3203 Homework 5 Due 10 Feb 2021

This homework is due to `https://uwcloud.uwinnipeg.ca/s/T6ykcP988pa3kpG` by 10:59PM on the due date. You may submit a PDF either scanned from handwriting or generated with LaTeX or a word processor (with an equation editor).

1. **Box on a Wedge Hamiltonian Version**

   Consider again the box of mass m sliding down a wedge of mass M on a frictionless horizontal surface. See the figure

   On the previous assignment, you should have found that the Lagrangian is

   $$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 - 2\dot{x}\dot{y}\cos\alpha\right) + mgy\sin\alpha \ . \tag{1}$$

   (a) Show that the velocities and canonical momenta are related by

   $$p_x = (M+m)\dot{x} - m\dot{y}\cos\alpha \ , \quad p_y = m(\dot{y} - \dot{x}\cos\alpha) \tag{2}$$

   and

   $$\dot{x} = \frac{p_x + p_y\cos\alpha}{M + m\sin^2\alpha} \ , \quad \dot{y} = \frac{p_y}{m} + \frac{p_x + p_y\cos\alpha}{M + m\sin^2\alpha}\cos\alpha \ . \tag{3}$$

   (b) Find the Hamiltonian. *Hint:* note that the term in parenthesis in the Lagrangian can be written $p_y^2/m^2 + \dot{x}^2\sin^2\alpha$.

   (c) Name two conserved quantities in this system.

2. **Hamiltonian Central Force Motion** *expanded from Kibble & Berkshire*

   Consider an object of mass m moving in 3D with a central conservative force of potential energy $V(r)$.

   (a) Write the Hamiltonian for this object in spherical polar coordinates.

   (b) You should see that the azimuthal angle ϕ is cyclic. Assuming motion is confined to the equatorial plane, find the effective potential for radial motion. Find the transformation of the Cartesian coordinates generated by p_ϕ. Use both these results to argue that $p_\phi = J_z$, the z component of angular momentum. *Hint:* for the transformation, write the Cartesian coordinates in terms of polar coordinates. Then compare to the transformation from the class notes.

   (c) Define the square angular momentum

   $$\vec{J}^2 = m^2 r^4\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) \ . \tag{4}$$

   Write $\vec{J}^2$ in terms of canonical momenta and show that it is conserved, even though θ is not cyclic. *Hint:* what is the Poisson bracket with the Hamiltonian?

3. **Liouville Theorem in Particle Accelerators** *from Thornton & Marion*

Consider a linear accelerator, which accelerates bunches of electrons along the z axis. The beam initially has a circular cross section of radius R in the xy plane with uniform electron density across the circle. The transverse momenta p_x, p_y are likewise distributed uniformly over a circle of radius P centered on the origin of momentum space. As they move down the accelerator, some mechanism focuses the beam onto a transverse circle in the xy plane of half the initial radius. What is the resulting distribution of p_x, p_y, and what does this mean for the electron beam? (This should be a short answer that does not require detailed calculation.) *Hint:* The z motion is decoupled from the motion in the xy plane, so you can focus only on the distribution in xy phase space.