PHYS-3203 Homework 3 Due 27 Jan 2020

This homework is due to `https://uwcloud.uwinnipeg.ca/s/T6ykcP988pa3kpG` by 10:59PM on the due date. You may submit a PDF either scanned from handwriting or generated with LaTeX or a word processor (with an equation editor).

1. **Sliding on a Cycloid**

   Consider a cycloidal track (like we found for the brachistochrone)

   $$x = a(\theta - \sin\theta)\ , \quad y = a(1 - \cos\theta) \tag{1}$$

   with y increasing downward. The coordinate θ extends from $\theta = 0$ at the left of the track $x = 0, y = 0$ to $\theta = 2\pi$ at the right $x = 2\pi a, y = 0$, and the lowest point of the track is $x = a\pi, y = 2a$ at $\theta = \pi$.

   (a) Write the Lagrangian and Euler-Lagrange equation for an object sliding frictionlessly on the cycloid in terms of the generalized coordinate θ. Show that the equation of motion is that of a harmonic oscillator for $\theta = \pi + u$ where $|u| \ll 1$ (ie, small amplitude motion around the lowest point of the cycloid).

   To learn more about motion on the cycloid, try a different generalized coordinate. Define

   $$s = \int_\pi^\theta d\theta' \sqrt{\left(\frac{dx}{d\theta'}\right)^2 + \left(\frac{dy}{d\theta'}\right)^2}\ , \tag{2}$$

   so $|s|$ is the distance traveled from the lowest point along the cycloid.

   (b) Find the kinetic energy in terms of s. *Hint:* Use the infinitesimal form of the Pythagorean theorem to find $\dot{s}^2$.

   (c) Write the potential energy in terms of s. *Hint:* Using a half-angle formula, integrate (2) to get s in terms of θ. Then use the angle addition formula to get s^2 in terms of y.

   (d) Using your results for the kinetic and potential energies, write the Lagrangian in terms of s. Then write the Lagrangian for a simple harmonic oscillator of frequency ω. By comparison of the two Lagrangians, show that motion on the cycloid is simple harmonic and give the frequency (which is independent of amplitude). You do not need to find the Euler-Lagrange equations.

2. **Falling Ladder** *from Kibble & Berkshire*

   A straight ladder of length $2L$ leans against a wall — one end is on the floor $y = 0$, and the other is on the wall $x = 0$. Both the wall and floor are frictionless. The ladder is symmetric, so its center of mass is at a distance L from either end, and it has mass M and moment of inertia I around the axis perpendicular to the xy plane and through the center of mass.

   (a) Find the position of the center of mass as a function of the angle θ of the ladder from the horizontal, assuming the end of the ladder is still touching the wall. Then write a Lagrangian for the motion of the ladder and the Euler-Lagrange equation in terms of the generalized coordinate θ. *Hint:* remember that the kinetic energy can be written as translational kinetic energy of the center of mass plus rotational kinetic energy around the center of mass.

(b) Because the motion is frictionless, energy is conserved. Using the kinetic energy and potential energy you found above, write an energy conservation equation for the generalized coordinate θ using θ_0 as the initial position of the ladder, where it is instantaneously at rest.

(c) Now write the Lagrangian for the system by implementing the relationship between the center of mass positions x, y and θ with Lagrange multipliers. Find the equations of motion.

(d) With initial conditions that the ladder is at rest at some angle θ_0, at what angle θ does it lose contact with the wall? Recall that the Lagrange multiplier for horizontal position gives the force of constraint and vanishes when the ladder leaves the wall. *Hint:* use the EOM for x re-written in terms of θ and then use the EOM and energy conservation you found in (a).