Systems of Particles

General Principles

- Relativity Principle (reminder)
  - Physics is the same in all inertial reference frames, ie, forces in Newton's laws have a physical origin
  - We can choose a convenient reference frame to work in. We're now going to mostly pay attention to the relation between a general frame + the rest frame of ~~the~~ center of mass of an object or group of particles.

- Reminder on changing frames: Simple Procedure
  - Work out change of position coordinates
    Ex $\vec{r}' = \vec{r} + \vec{v}t$ for prime frame moving at $\vec{v}$
  - Then differentiate, etc to find transformation of other variables
  - Should be familiar from last term

The Center of Mass Frame for Many Particles

- Center of Mass (CM) and Momentum
  - Suppose we have a set of particles of masses m_i and positions $\vec{r}_i$
    - + The total mass $M = \sum_i m_i$
    - + The center of mass position is the mass weighted average
      $$\vec{R} = \left(\sum_i m_i \vec{r}_i\right)/M$$
    - + We can convert these to integrals for continuum materials

  - Total momentum
    - + Is $\vec{P} = \sum m_i \dot{\vec{r}}_i = M\dot{\vec{R}}$ = momentum of a particle mass M moving like the CM
    - + Also,
      $$\dot{\vec{P}} = \sum_i \left(\vec{F}_{i,ext} + \sum_j \vec{F}_{ij}\right) = \vec{F}_{ext} + \sum_{ij} \vec{F}_{ij} = \vec{F}_{ext}$$
      total external force

      b/c $\vec{F}_{ij} = -\vec{F}_{ji}$ by 3rd law.

+ In the absence of forces external to the system, this is conservation of momentum

+ Suppose the external forces are due to uniform gravity

$$\vec{F}_{i,ext} = m_i \vec{g} \Rightarrow \dot{\vec{P}} = M\vec{g}$$

The CM moves like a single particle in gravity.

- Angular Momentum
  - Define $\vec{r}_i = \vec{R} + \vec{r}_i^*$ where $\vec{r}_i^*$ = position relative to CM, ie, position in CM frame
  - Then we re-write $\vec{J}$

+ $\vec{J} = \sum \left[m_i \vec{R} \times \dot{\vec{R}} + m_i \vec{r}_i^* \times \dot{\vec{R}} + m_i \vec{R} \times \dot{\vec{r}}_i^* + m_i \vec{r}_i^* \times \dot{\vec{r}}_i^* \right]$

+ But $\sum m_i \vec{r}_i^*$ = CM position in CM frame $= 0$

so

$$\vec{J} = M \vec{R} \times \dot{\vec{R}} + \vec{J}^*$$

= "orbital angular momentum" of CM + angular momentum around the CM ($\vec{J}^*$)

- Consider $\dot{\vec{J}} = \sum m_i \vec{r}_i \times \ddot{\vec{r}}_i$

+ From Newton's law

$$\dot{\vec{J}} = \sum_i \left(\vec{r}_i \times \vec{F}_{i,ext} + \sum_j \vec{r}_i \times \vec{F}_{ij} \right)$$

where $\vec{F}_{ij}$ = force on i from j

+ But

$$\sum_{i,j} \vec{r}_i \times \vec{F}_{ij} = \frac{1}{2} \sum (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij} \quad \text{by } 3^{rd} \text{ law}$$

if $\vec{F}_{ij}$ is central, this vanishes

+ In this case

$$\dot{\vec{J}} = \vec{\tau}_{ext} \quad \text{(external torque)}$$

+ But in fact we know from Noether's theorem that this must be true even for noncentral forces as long as the internal potential is rotationally symmetric. This now includes magnetic forces.

- What if the CM is accelerated?
  + The derivative of the "orbital" angular momentum is
$$M\vec{R}\times\ddot{\vec{R}} = \vec{R}\times\vec{F}_{ext}$$
  + Therefore,
$$\dot{\vec{J}}^* = \dot{\vec{J}} - \vec{R}\times\vec{F}_{ext} = \sum(\vec{r}_i-\vec{R})\times\vec{F}_{i,ext} = \sum \vec{r}_i^*\times\vec{F}_{i,ext}$$
  + This is the torque around the CM position generated by physical and not fictitious forces even if the CM is accelerating. This works out b/c $\vec{R}^*=0$

— Energy
- Kinetic energy also splits into CM + CM frame parts
$$T = \frac{1}{2}\sum m_i\dot{\vec{r}}_i^{\,2} = \frac{1}{2}M\dot{\vec{R}}^2 + T^*, \quad T^* = \frac{1}{2}\sum m\dot{\vec{r}}_i^{*2}$$
- The time derivative is
$$\dot{T} = \sum_i \dot{\vec{r}}_i\cdot\vec{F}_{i,ext} + \sum_{i,j}\dot{\vec{r}}_i\cdot\vec{F}_{ij}$$
  + The internal forces are $2\sum_{i<j}(\dot{\vec{r}}_i-\dot{\vec{r}}_j)\cdot\vec{F}_{ij}$
  + In a rigid body, $|\vec{r}_i-\vec{r}_j|^2 = \text{const} \Rightarrow (\vec{r}_i-\vec{r}_j)\cdot(\dot{\vec{r}}_i-\dot{\vec{r}}_j)=0$ so a central internal force does no work.
  + If $\vec{F}_{ij}$ is conservative, relativity principle means it is central and arises from $V_{ij}(|\vec{r}_i-\vec{r}_j|)$
    We can define an internal potential
$$V_{int} = \sum V_{ij}$$
    and
$$\frac{d}{dt}(T+V_{int}) = \sum\dot{\vec{r}}_i\cdot\vec{F}_{i,ext}$$
  + If external forces are conservative, there is a conserved total energy $T+V_{int}+V_{ext}$
- CM frame energy
  + Even if the CM is accelerated, we can see
$$\frac{d}{dt}(T^*+V_{int}) = \sum\dot{\vec{r}}_i^*\cdot\vec{F}_{i,ext}$$
    for physical external forces. Again, fictitious forces don't contribute.

+ For motion in uniform gravity,

$$T = \tfrac{1}{2} M \dot{\vec{R}}^2 + T^*, \qquad V = V_{int} - \sum m_i \vec{r}_i \cdot \vec{g} = V_{int} - M\vec{R}\cdot\vec{g}$$

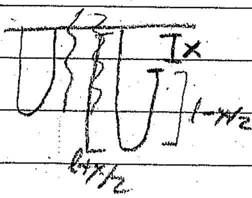
$$\Rightarrow \quad L = \left(\tfrac{1}{2} M \dot{\vec{R}}^2 + M\vec{g}\cdot\vec{R}\right) + \left(T^* - V_{int}\right)$$

so motion separates into motion of CM and motion around CM (as we've argued before). This means there is no net external torque also. Direct calculation is

$$\vec{\tau}_{ext} = \sum \vec{r}_i^{\,*} \times \vec{F}_{i,ext} = \left(\sum m_i \vec{r}_i^{\,*}\right) \times \vec{g} = 0.$$

— Examples: "Floppy Body" Motion

A few cases where we can use these ideas for motion of non-rigid objects



- Hanging String: A string of length $2l$ and linear mass density μ is attached at one end to the ceiling. The other end is initially at the ceiling, then falls.

+ Suppose the right-hand end has fallen distance x. If we ignore any horizontal swinging, we just need to understand the CM motion. The CM position is

$$X = \left[m_{left}\,(l + x/2)/2 + m_{right}\,(x + (l - x/2)/2)\right]/2\mu l$$

with $m_{left} = \mu(l + x/2)$, $m_{right} = \mu(l - x/2)$

$$\Rightarrow \quad X = (l + x - x^2/4l)/2$$

+ The total momentum (downward) is therefore

$$\vec{P} = (2\mu l)\dot{X} = \mu(l - x/2)\dot{x}$$

+ External forces are gravity and tension on the left end, so

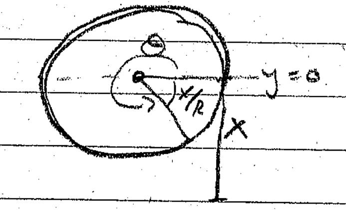
$$\dot{\vec{P}} = \mu(l - \tfrac{x}{2})\ddot{x} - \tfrac{\mu}{2}\dot{x}^2 = (2\mu l)g - T$$

If we assume that there is no tension at the free end, x is in free fall, so $\ddot{x} = g$, $\dot{x} = \sqrt{2gx}$. Then

$$T = \mu l g + 3\mu g x/2$$

+ There are actually a number of ways a falling chain behaves differently — horizontal motion is important.

• Unwinding Rope: A rope of linear density μ wraps once around a disk or cylinder of radius R. The disk can rotate around its center to unwind the rope



+ The vertical center of mass of the rope is the mass-weighted avg of the vertical segment CM + the wound segment CM

$$Y = \frac{1}{2\pi R\mu}\left\{(\mu x)\left(-x/2\right) + \mu R^2 \int_0^{2\pi - x/R} d\theta \, \sin\theta \right\}$$

so potential energy

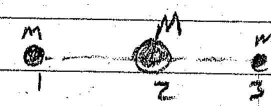
$$V = -MYg = g\left[x^2/2R + \cos(x/R) - 1\right]$$

* The kinetic energy is due to the speed of the rope, which has speed $\dot{x}$ all along, and the rotation of the disk with $\omega = \dot{x}/R$. So

$$T = \tfrac{1}{2}\mu(2\pi R)\dot{x}^2 + \tfrac{1}{2} I \dot{x}^2/R^2$$

+ Conservation of energy gives speed as function of position, etc

• Linear Triatomic Molecule



Take a molecule w/ 3 atoms in a line (in equilibrium). The middle has mass M, other 2 mass m. Want to describe types of motion

+ First, we can work in CM frame. There can be rotational motion (ie, rigid body) around CM and also small vibrations
+ In CM frame, $\vec{r}_2 = -(m/M)(\vec{r}_1 + \vec{r}_3)$
Purely vibrational motion has $\vec{J} = 0$.
+ The remaining vibrational motion has normal modes (in the xy plane)

$$x_1 = -x_3, \quad x_2 = 0$$

$$x_1 = x_3, \quad x_2 = -2m x_1/M$$

longitudinal

and

$$y_1 = y_3, \quad y_2 = -2m y_1/M$$

transverse

- Two Particles
  - There is a further simplification when there are only 2 objects.
  - The CM frame kinetic energy is
    $$T^* = \frac{1}{2} m_1 \dot{\vec{r}}_1^{*2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{*2}$$
    + Define the relative position $\vec{r} = \vec{r}_1^* - \vec{r}_2^*$
    This is actually also $\vec{r} = \vec{r}_1 - \vec{r}_2$ for any frame related by shift of origin or velocity
    + We also know $m_1 \vec{r}_1^* + m_2 \vec{r}_2^* = 0$, so
    $$\vec{r}_1^* = m_2 \vec{r}/M, \quad \vec{r}_2^* = -m_1 \vec{r}/M$$
    + Simplifying
    $$T^* = \frac{1}{2}\left(\frac{m_1 m_2^2 + m_2 m_1^2}{M^2}\right) \dot{\vec{r}}^2 = \frac{1}{2} \mu \dot{\vec{r}}^2$$
    where the reduced mass is
    $$\mu = m_1 m_2 / M \quad (\text{and is always } \mu < m_1, m_2)$$
    + This is like a single particle b/c the motion of the 2nd particle "mirrors" the 1st.
  - Also, consider the angular momentum
    + $\vec{J}^* = m_1 \vec{r}_1^* \times \dot{\vec{r}}_1^* + m_2 \vec{r}_2^* \times \dot{\vec{r}}_2^* = \mu \vec{r} \times \dot{\vec{r}}$
    + Again, we have the angular momentum of a single effective particle of mass μ following the path $\vec{r}$.
  - We will look at consequences + applications next.

- Applications
  - Two-Body Collisions Dynamics: CM vs

    We want to understand the relation between the CM frame + lab frame descriptions of a collision

    ∘ We will consider elastic collisions for simplicity

    • CM frame description

    + Elastic because kinetic energy is conserved. That means (1) no energy loss to heat (2) forces between objects → 0 at large distance ($V \to$ constant)
    + The kinetic energy is (in terms of relative position)

      $$T = \tfrac{1}{2}\mu \dot{\vec{r}}^{\,2} = \vec{p}^{*2}/2\mu, \quad p^* \equiv \mu\dot{\vec{r}} = m_1\dot{\vec{r}}_1^* = -m_2\dot{\vec{r}}_2^*$$

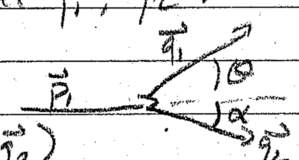
      Call $\vec{p}^*$ the CM frame momentum (of 1 object)
    + If $\vec{p}^*$ and $\vec{q}^*$ are initial + final CM frame momenta,

      $$|\vec{p}^*| = |\vec{q}^*| \quad \text{\underline{no matter} the forces}$$

    + The objects change direction only and not speed in CM frame
    + The scattering angle Θ^* is the deflection of each object and is the angle between $\vec{p}^*$ and $\vec{q}^*$

    • The (fixed target) lab frame

    + This is a common experiment design where object 2 is initially at rest
    + Define initial $\vec{p}_1, \vec{p}_2 = 0$ and final $\vec{q}_1, \vec{q}_2$ momenta
    + The 1st particle has scattering angle Θ (angle between $\vec{p}_1$ + $\vec{q}_1$) while the 2nd has recoil angle α (between $\vec{p}_1$ + $\vec{q}_2$)

    + It's often easier to work in CM frame + convert

    ∘ Converting frames.

    + We know that initially

      $$\vec{p}_1 = m_1\dot{\vec{r}}_1 = M\dot{\vec{R}} \quad \text{and} \quad 0 = \dot{\vec{r}}_2 = \dot{\vec{R}} + \dot{\vec{r}}_2^*$$

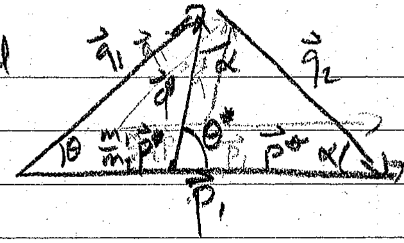
      $$\Rightarrow \vec{p}_1 = (M/m_2)\,\vec{p}^*, \quad \dot{\vec{R}} = \vec{p}^*/m_2$$

    + After the collision,

      $$\vec{q}_1 = m_1(\dot{\vec{R}} + \dot{\vec{r}}_1^*) = (m_1/m_2)\vec{p}^* + \vec{q}^* \quad \text{and}$$

      $$\vec{q}_2 = m_2(\dot{\vec{R}} + \dot{\vec{r}}_2^*) = p^* - q^*$$

+ We can therefore construct nested triangles of the momenta. The scattering + recoil angles sit at same vertices



+ The recoil angle appears at 2 vertices b/c $|\vec{p}^*| = |\vec{q}^*|$
This tells us that $\alpha = \frac{1}{2}(\pi - \theta^*)$ and
$|\vec{q}_2| = 2|\vec{p}^*| \sin(\frac{1}{2}\theta^*)$ (by dropping a $\perp$)

+ The fractional energy transfer in lab frame is the KE of the final 2nd particle relative to the total, or

$$\left(|\vec{q}_2|^2/2m_2\right) / \left(|\vec{p}_1|^2/2m_1\right) = \frac{\left(2|\vec{p}^*|^2 \sin^2(\theta^*/2)\right)/m_2}{M^2|\vec{p}^*|^2/2m_1 m_2^2}$$

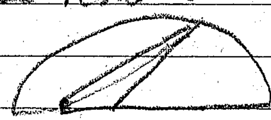
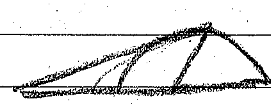
$$= \left(4m_1 m_2/M^2\right) \sin^2(\theta^*/2)$$

The max occurs when $\theta^* = \pi$, particles reversed in CM frame. It only $\to 1$ when $m_1 = m_2$ (Newton's cradle)

+ Meanwhile, dropping a $\perp$ from the top vertex relates the scattering angles

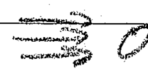
$$\tan\theta = \sin\theta^* / (\cos\theta^* + m_1/m_2)$$

+ Max lab scattering angle:
If $m_1 < m_2$, $\theta \to \pi/2$ when $\cos\theta^* = -m_1/m_2$, then increases to π
But if $m_1 > m_2$, we can never have $\theta = \pi/2$. Instead, θ increases to θ_{max}, then decreases back to 0. With a little work, we see $\sin\theta_{max} = m_2/m_1$
These results are encapsulated in the diagrams

 and 

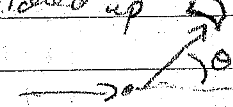
− Cross Sections in Different Frames

• Review of definitions
+ We imagined a flux f of incoming particles (ie, #/time) hitting a target of cross sectional area σ, so #collisions/time $= f\sigma$.



+ This lets us define generally $\sigma \equiv$ (collisions/time)$/f$.

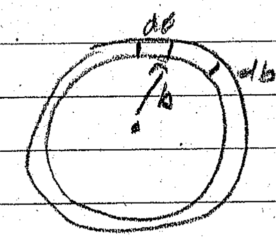
+ The number of scattering events that are picked up by a detector of area dA at distance R from the target is



$$\text{scatters/time} = f \left(\frac{d\sigma}{d\Omega}\right) dA/R^2$$

+ This defines differential cross section $d\sigma/d\Omega$.
If we think of the sphere of radius R in terms of polar angles, the solid angle of dA is $d\Omega = dA/R^2 = \sin\theta \, d\theta \, d\phi$

+ The polar angle θ is the scattering angle, which is determined by impact parameter b, typically monotonically decreasing.



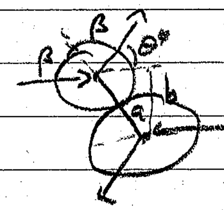
in CM frame + in lab frame for $m_1 < m_2$

Then $d\sigma = b\,|db|\,d\phi = (d\sigma/d\Omega)\,d\Omega$.

- Lab vs CM Frame Cross Sections
  + Take a lab frame with target particles initially at rest. The total targets is $n_2 V$, where n_2 = number density, V = volume, and total flux is $n_1|\vec{v}|$, where n_1 = number density + $\vec{v}$ = velocity of initial incoming particles. So total rate of scatterings into dA is $dw = n_1 n_2 |\vec{v}| V (d\sigma/d\Omega)\, dA/R^2$
  + But in CM frame, the particles move past each other at a rate given by the relative velocity $\Delta\vec{v} = \vec{v}_1 - \vec{v}_2$. The scattering rate is
  $$dw = n_1 n_2 |\Delta\vec{v}| V (d\sigma/d\Omega^*)\, dA/R^2$$
  but the area is oriented differently

  + It is often easier to get the lab cross section by calculating the CM cross section and converting
  $$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega^*}\left(\frac{\sin\theta^* \, d\theta^*}{\sin\theta \, d\theta}\right) = \frac{d\sigma}{d\Omega^*}\frac{d\cos(\theta^*)}{d\cos(\theta)}$$
  We can get the conversion factor from
  $$\tan\theta = \sin\theta^* / (\cos\theta^* + m_1/m_2)$$
  A simple case is $m_1 = m_2 \Rightarrow \theta^* = 2\theta$, so
  $$\frac{d\cos\theta^*}{d\cos\theta} = \frac{d}{d\cos\theta}(2\cos^2\theta - 1) = 4\cos\theta.$$
  + Some complication in this conversion for $m_1 > m_2$ b/c of θ_{max}

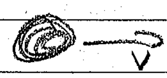
- Example: Hard Sphere Scattering
  + Take 2 spheres of radii a_1 + a_2, masses m_1 + m_2. They have an elastic contact interaction, so they scatter whenever the impact parameter $b \le a \equiv a_1 + a_2$. The total cross section is $\sigma = \pi a^2$
  + Differential cross section determined by realizing scattering is reflection from contact plane. As per figure, this means $\theta^* = \pi - 2\beta$

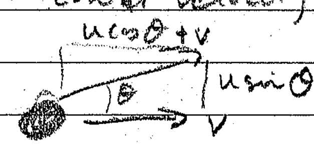


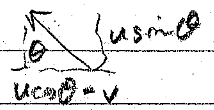
  + Then we have
$$b = a \sin\beta = a\cos(\theta^*/2)$$
so
$$b\,|db| = (a^2/2)\cos(\theta^*/2)\sin(\theta^*/2)\,d\theta^* = (a^2/4)\sin\theta^*\,d\theta^*$$
and
$$d\sigma/d\Omega^* = a^2/4 .$$
Same as when $m_2 \to \infty$ from the fall.
  + But what about lab frame with m_2 at rest initially? Take $m_1 = m_2$ for simplicity. Then $\theta < \pi/2$ — no backwards scattering. With our conversion
$$d\sigma/d\Omega = a^2\cos\theta .$$
This also integrates to give $\sigma = \pi a^2$.

— Orbital Mechanics

- Gravity Assist: A space probe approaches a planet with relative speed u.
  + Since the planet is effectively infinitely massive, CM frame is planet rest frame. Probe leaves w/ relative speed u.
  + In solar frame, the probe initial "horizontal" velocity is $(u\cos\theta - v)$ for planetary velocity v. After, horizontal velocity of probe is $u\cos\theta + v$.







Increase of twice planetary orbit speed!

- Elliptical orbits
  + The actual Lagrangian is
$$L = \tfrac{1}{2}\mu\dot{r}^2 + GM\mu/r \quad \text{since } m_1 m_2 = \mu M.$$

+ Therefore, the orbit is the same as an object of mass μ orbiting a fixed center of mass M = total mass.

+ In our solar system, we almost always have $m_2 \gg m_1$, so $M \to m_2$, $\mu \to m_1$. Earth-moon is a slight exception

+ There is a slight change to Kepler's 3rd law:

$$(T/2\pi)^2 = a^3/GM$$

, where a = semi-major axis for relative position and M = total mass.

This changes slightly for each planet (a moon)

+ Each orbiting object orbits the center of mass in an ellipse with $\vec{r}_1^* = m_2 \vec{r}/M$, $\vec{r}_2^* = -m_1 \vec{r}/M$.

That means the semi-major axes are

$$a_1 = m_2 a/M, \quad a_2 = m_1 a/M.$$

- Tidal Friction

+ Think of the high tides as bulges of water. These want to rotate w/ the earth, but the tidal forces of the moon pull them back toward the moon. This tends to slow down the rotation

+ Quantitatively, the CM frame angular momentum is

$$\vec{J}^* = \mu \vec{r} \times \dot{\vec{r}} + \vec{J}_\oplus^* + \vec{J}_m^*$$

where $\vec{J}_\oplus + \vec{J}_m$ are due to earth's + moon's rotations, $\vec{J}_m$ is smaller than $\vec{J}_\oplus$ and the orbital term

+ Angular momentum conservation is

$$J^* = \mu a^2 \Omega + I_\oplus \omega$$

where a = orbit semimajor axis, Ω = orbit frequency, $I_\oplus$ = earth moment of inertia, ω = rotation frequency of earth.

+ But we further know

$$\Omega^2 a^3 = GM \quad \text{(Kepler's 3rd law)}$$

so

$$J^* = \mu \sqrt{GMa} + I_\oplus \omega$$

+ So, as the tidal friction slow earth's rotation, the orbit semi-major axis must decrease. This is a transfer of energy from earth to orbit.

- Restricted 3-body Problem

+ In general, it's impossible to solve the motion of 3 objects interacting gravitationally except on a computer. This is the "3-body problem". This might describe the sun, earth, & moon, a planet around binary stars, etc. The more general N-body problem describes a solar system and (importantly) the formation of structure in the universe.

+ In the 3-body problem, call the objects the primary, secondary, and tertiary in order of most to least massive ($m_1 > m_2 > m_3$)

+ To make progress, we will look at the restricted 3-body problem:
  1) We assume the tertiary is very light $m_3 \ll m_1, m_2$ so its gravity does not meaningfully affect primary or secondary
  2) All motion is in one plane
  3) The primary & secondary orbit their CM circularly

+ Work in a frame rotating with the primary & secondary and their CM at the origin. They are located at $a_1 = m_2 a/(m_1+m_2)$ and $-a_2 = -m_1 a/(m_1+m_2)$ on the $\hat{x}$ axis. Angular velocity of the frame is $\vec{\omega} = \omega\hat{z}$ with $\omega^2 = G(m_1+m_2)/a^3$.

+ If the tertiary is at $\vec{r} = x\hat{x} + y\hat{y}$ in the rotating frame, it experiences force (including fictitious forces)

$$\vec{F} = -m\left(\frac{Gm_1}{r_1^3}\vec{r}_1 + \frac{Gm_2}{r_2^3}\vec{r}_2\right) - 2m\vec{\omega}\times\dot{\vec{r}} + m\omega^2\vec{r}$$

where

$\vec{r}_1 = \vec{r} - a_1\hat{x}$, $\vec{r}_2 = \vec{r} + a_2\hat{x}$ are the relative positions

+ There are 5 Lagrangian points where $\vec{F}$ vanishes on a stationary tertiary. L_1 to L_3 are on the x axis. L_1 is between m_1 & m_2, L_2 is past m_2, & L_3 is past m_1. These are unstable. L_4 & L_5 are the vertices of equilateral triangles with primary & secondary at the others. Due to Coriolis force, there are stable small motions near L_4 & L_5.