Lagrangian + Hamiltonian Mechanics

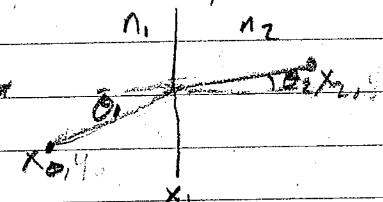
We are going to learn 2 powerful new ways of thinking about classical mechanics (+ solving problems)

Calculus of Variations

We want to answer questions like "What function minimizes some particular quantity?" For example, if you have a chain of fixed length hanging from 2 fixed points, what shape minimizes its potential energy?

– Motivating Exercise: Snell's Law + Fermat's Principle

- We remember Snell's law that light leaving a region with index n_1 and entering one of n_2 satisfies $n_1 \sin\theta_1 = n_2 \sin\theta_2$



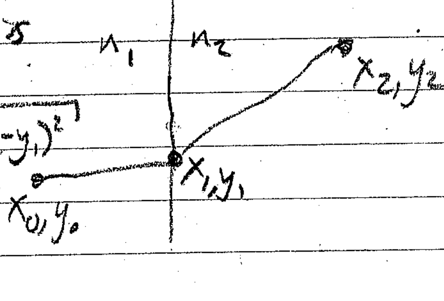
where $\theta_{1,2}$ are the angles of incidence + refraction.

- Fermat's principle states that a light ray moving from point A to point B follows the path of least time between those points.

- Recalling that $n = c/v$ tells us the speed of light in the medium, we can derive Snell's law from Fermat's principle of least time.

  + We'll prove it later, but now assume that a line segment is the shortest distance between 2 points

  + The time from $A = (x_0, y_0)$ to $B = (x_2, y_2)$ passing through (x_1, y_1) on the boundary of the 2 regions is



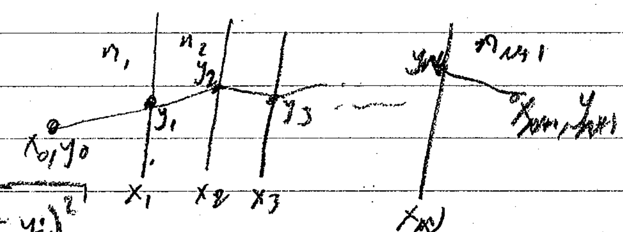
$$T = \frac{n_1}{c}\sqrt{(x_1-x_0)^2+(y_1-y_0)^2} + \frac{n_2}{c}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

+ We can minimize wrt y_1:

$$\frac{dT}{dy_1} = \frac{n_1}{c}\frac{y_1-y_0}{\sqrt{\cdots}} - \frac{n_2}{c}\frac{y_2-y_1}{\sqrt{\cdots}} = 0$$. Note: $y_0 + y_2$ fixed

In each case $\frac{\Delta y}{\sqrt{\Delta x^2+\Delta y^2}} = \sin\Theta$, so we get Snell's law.

• Now suppose we have a series of different materials stacked together



+ The total time is

$$T = \frac{1}{c}\sum_{i=0}^{N} n_i \sqrt{(x_{i+1}-x_i)^2 + (y_{i+1}-y_i)^2}$$

+ To minimize we need $\partial T/\partial y_j = 0$, $j = 1 \ldots N$
with $\partial y_i / \partial y_j = \delta_{ij}$

- More generally:

• If we set all $x_i - x_{i-1} = \Delta x$, we have a sum of the general form

$$S = \sum_i f(x_i, y_{i+1}-y_i) = \sum_i f(x_i, y_i')$$

+ Note that $$\partial S/\partial y_j = -\frac{\partial f}{\partial y_j'} + \frac{\partial f}{\partial y_{j-1}'}$$

+ More generally, we can have $S = \sum f(x_i, y_i, y_i')$

so $$\frac{\partial S}{\partial y_j} = \frac{\partial f}{\partial y_j} + \frac{\partial f}{\partial y_{j-1}'} - \frac{\partial f}{\partial y_j'} \qquad (*)$$

(In our example, this is the case if $n = n(x,y)$)

• What if $n_i \to n(x)$ varies continuously

+ Convert the sum to an integral

$$\cancel{T}\, S = \sum f = \sum \Delta x \,(f/\Delta x) \to \int_{x_0}^{x_f} dx\, F(x, y, y')$$

For our example $T = \frac{1}{c}\int_{x_0}^{x_f} dx\, n(x)\sqrt{1+y'(x)^2}$ w/ $y(x_0) \equiv y_0$ + $y(x_f) = y_f$ fixed

+ We say $S[y]$ (or T) is a functional of the independent variable x and the function $y(x)$ (and its derivative)

• What are the functional derivatives of S?

+ Comparing to (*), we should have

$$\frac{\delta S}{\delta y(x')} = \frac{\partial F}{\partial y}(x', y(x'), y'(x')) - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)(x', y(x'), y'(x'))$$

Treating F as a function of x, y, y' as separate variables

+ This works if we take $\delta y(x)/\delta y(x') = \delta(x-x')$ and $\delta y'(x)/\delta y(x') = \frac{d}{dx}(\delta(x-x'))$

Then

$$\frac{\delta S}{\delta y(x')} = \int dx \left\{ \frac{\partial F}{\partial y}\frac{\delta y(x)}{\delta y(x')} + \frac{\partial F}{\partial y'}\frac{\delta y'(x)}{\delta y(x')} \right\}$$

$$= \int dx \left\{ \frac{\partial F}{\partial y}\delta(x-x') + \underbrace{\frac{\partial F}{\partial y'}\frac{d}{dx}(\delta(x-x'))}_{\text{integrate by parts}} \right\}$$

$$= \int dx\, \delta(x-x')\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right) = \text{desired outcome.}$$

+ Then $\delta S/\delta y(x) = 0$ gives the Euler-Lagrange eqn (or Euler eqn) $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial y}$ to optimize S

+ In our example,

$$\frac{\delta T}{\delta y(x)} = 0 \Rightarrow \frac{d}{dx}\left[n(x) \frac{y'(x)}{\sqrt{1+y'(x)^2}} \right] = 0.$$

Suppose $n(x) = \text{constant}$. Then $y'(x) = C\sqrt{1+y'(x)^2}$ for $C = \text{const}$. That gives $y'(x)^2 = \text{constant}$, or $y = mx + b$ = straight line.

In other words, straight lines are the shortest paths between 2 specified points.

- An alternate approach to functional derivatives:
  + We want to find the function $y(x)$ that minimizes S. So suppose $y(x)$ is the optimum one.
  + What is $S[y+\delta y]$? Assume $y(x_0), y(x_f)$ fixed, so $\delta y \to 0$ at endpts
  + $$S[y+\delta y] = \int_{x_0}^{x_f} dx\, F(x, y+\delta y, (y+\delta y)')$$

    ↑ Taylor expansion

$$= \int_{x_0}^{x_f} dx \left\{ F(x,y,y') + \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' \right) + \dots \right\}$$

For optimization, we ignore higher-order ... terms (but see reading)

+ The change in $y' = dy/dx$ is $\delta y' = d\delta y/dx$,

so
$$\int_{x_0}^{x_f} dx\, \frac{\partial F}{\partial y'}\delta y' = \int_{x_0}^{x_f} dx\, \frac{\partial F}{\partial y'} \frac{d\,\delta y}{dx}$$
$$= \left[\frac{\partial F}{\partial y'}\delta y \right|_{x_0}^{x_f} - \int_{x_0}^{x_f} dx\, \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y$$

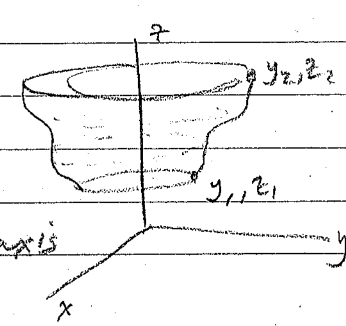
The bdry terms vanish b/c $\delta y(x_0) = \delta y(x_f) = 0$

+ Then at 1st order in δy
$$\delta S = \int_{x_0}^{x_f} dx \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \right] \delta y$$

for any δy that does not change the Dirichlet b.c. on $y(x)$

Thus, we must have:
$$\frac{\delta S}{\delta y} = \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \text{Euler-Lagrange.}$$

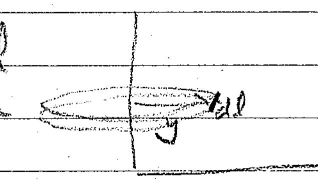
- Examples:



- Minimal surface of revolution:
  2 circular loops are centered on z axis

+ parallel to xy plane. In yz plane, 1st passes through (y_1, z_1) and 2nd through (y_2, z_2). Is the minimal area surface of revolution extending from 1 loop to the other? (This is the surface a soap film would take)

+ The area of a strip of width dl passing through (y, z) in the + yz plane is



$$dA = 2\pi y \, dl$$

+ Therefore if $z(y)$ is the curve in that quadrant from (y_1, z_1) to (y_2, z_2), the area to minimize is

$$A = 2\pi \int_{y_1}^{y_2} y \sqrt{1 + (dz/dy)^2} \, dy$$

b/c $dl^2 = \sqrt{dy^2 + dz^2}$ by Pythagorean theorem

+ Since we've taken y as the independent variable, the Euler-Lagrange equation is

$$\frac{d}{dy}\left[\frac{y z'}{\sqrt{1+(z')^2}}\right] = 0$$

+ To solve, quantity $= [\ -\ ] = a = \text{const}$. Solving,

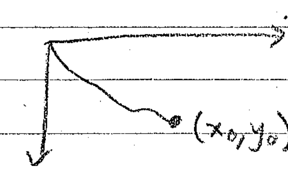
$$z = \int \frac{a \, dy}{\sqrt{y^2 - a^2}} = a \int d\theta \quad \text{with } y = a\cosh\theta$$

so

$$z = a\cosh^{-1}(y/a) + b, \quad y = a\cosh\left(\frac{z-b}{a}\right)$$

This is also the catenary, the equilibrium shape of a hanging chain.

- Brachistochrone ("shortest time") problem: An object slides down a track from $(0,0)$ to (x_0, y_0) under gravity. What is the shape of the track that minimizes the travel time? Assume no friction + starting from rest



+ With +y pointed down, the speed is $\sqrt{2gy}$, so the time to travel length dl is $dl/\sqrt{2gy}$.

+ The time to minimize is $T = \frac{1}{\sqrt{2g}}\int dy \sqrt{\frac{1+(x')^2}{y}}$ with y as independent variable

+ Euler-Lagrange eqn $\frac{d}{dy}\left[\frac{x'}{\sqrt{y(1+(x')^2)}}\right] = 0$

+ I'll just give the solution parametrically:

$$y = a(1-\cos\theta), \quad x = a(\theta - \sin\theta)$$

This path is a cycloid, or the position of a point on a circle rolling along x.

— Multiple Variables + Constraints

- If there are multiple independent variables $y_1(x), \ldots y_n(x)$, we have an independent Euler-Lagrange eqn to solve for each

$$\delta S/\delta y_1(x) = 0, \ \ldots \ \delta S/\delta y_n(x) = 0$$

- Sometimes there will be a constraint relating different coordinates
  + Example: indep. variable time t, dependent variables x, y, z for an object moving on a sphere. Then we can replace x, y, z by θ, ϕ angles of polar coords

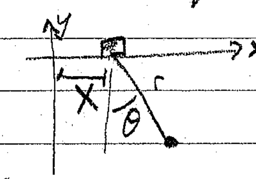
  + When we can solve the constraints + reduce the # of dependent variables (like above), the constraints are holonomic. In mechanics, the remaining dep. variables are the degrees of freedom.

  + Constraints may just involve the coordinates (like our sphere example) but could involve time derivatives. If the solution (the deg. of freedom) do not contain time explicitly, the system is natural.

If time appears explicitly, it is forced. We will normally have natural systems

- In order to solve holonomic constraints (or sometimes for convenience), we can use generalized coordinates

  + These aren't necessarily Cartesian or even polar (but could be)
  + In the context of mechanics, they are often denoted $q_1(t), q_2(t), \ldots$ with time derivatives $\dot{q}(t)$
  + Example: pendulum on moving support



This is a pendulum bob moving in xy plane with fixed distance r from a support that moves on the x axis

If the pendulum is at angle θ from vertical and the support is at X from the origin (generalized coords), we have the relation

$$x = X + r\sin\theta, \quad y = -r\cos\theta$$

- Lagrange Multipliers (aka undetermined multipliers)

  + What if constraints are non-holonomic? Can't just solve them

  + The example is a ball rolling (w/o slipping) on a surface. If the constraint were holonomic, the orientation of the ball would be determined by its position on the surface. But you can see that's not true.

  + Suppose you want to minimize $S = \int dx\, F(x, \{y\}, \{y'\})$ with some b.c. subject to the constraints $g_i(x, \{y\}, \{y'\}) = 0$.

  + Change the problem to minimizing

$$S' = \int dx \left[F(x, \{y\}, \{y'\}) + \sum_i \lambda_i(x)\, g_i(x, \{y\}, \{y'\}) \right]$$

where the λ_i are unknown functions called Lagrange multipliers

+ The Euler-Lagrange eqns of the λ_i are the constraints $g_i = 0$.

+ The Euler-Lagrange eqns of the y coordinates now include the λ_i, so we have to solve for more variables (but we have more equations)

+ Example: A disk rolls w/o slipping on a line. It therefore satisfies $x = R\theta$ where R = radius. We could use 1 generalized coordinate or else add $\lambda(t)(x(t) - R\theta(t))$ to our minimization problem.