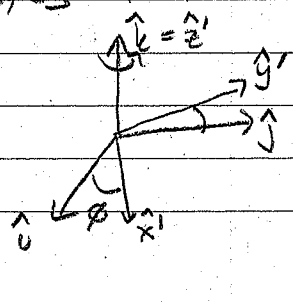
Rigid Body Rotation in Lagrangian + Hamiltonian Mechanics

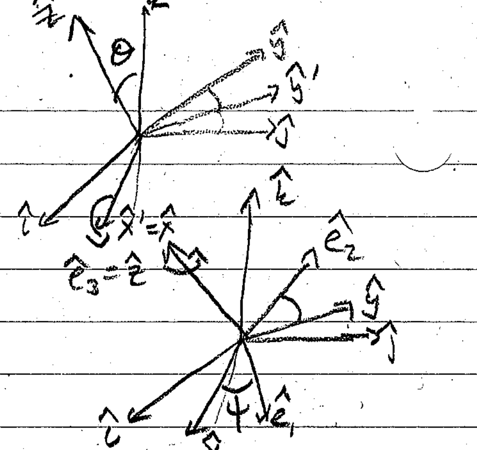
- Brief review
  - There are different ways to choose coordinate axes
    - + Fixed inertial axes. These don't move. We will denote them $\hat{i}, \hat{j}, \hat{k}$ etc
    - + Generally rotating axes. Denote $\hat{x}, \hat{y}, \hat{z}$, etc
    - + Body axes for a rigid object. These are rotating axes that are stationary w.r.t. the rotating rigid object
  - Moment of inertia tensor
    - + Defined as

$$I_{ij} = \int dm \left[r^2 \delta_{ij} - r_i r_j \right]$$

    - + Should calculate I_{ij} wrt body axes. Otherwise it changes in time wrt inertial axes as the object rotates
    - + The principal axes $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are the e'vectors of I_{ij} and are body axes. The e'values I_1, I_2, I_3 are principal moments of inertia

- Orientation of a rigid body
  - We need 3 angles as generalized coordinates to describe the orientation of an object wrt inertial axes.
    - + Most popular set of angles is due to Euler
    - + Another alternative is quaternions (generalized complex numbers) ← not spherical polars!
  - Euler (or Eulerian) Angles ϕ, θ, ψ
    - + Start w/ fixed inertial axes of unit vectors $\hat{i}, \hat{j}, \hat{k}$ and line up principal axes $\hat{e}_1, \hat{e}_2, \hat{e}_3$ with them
    - + Rotate the object by ϕ around $\hat{k}$. The principal axes now line up with $\hat{x}', \hat{y}', \hat{z}' = \hat{k}$





+ Next rotate by θ around $\hat{x}'$ to align principal axes with $\hat{x}=\hat{x}'$, $\hat{y}$, $\hat{z}$ axes

+ Finally rotate by ψ around $\hat{z}$ to get the principal axis positions. Note $\hat{z} = \hat{e}_3$

+ There are 3 important axes for this definition: $\hat{k}$, $\hat{x}=\hat{x}'$, $\hat{e}_3$. The $\hat{x}$ axis is the <u>line of nodes</u>. Note: our text and some others use $\hat{x}$ as the line of nodes. Some others use $\hat{y}_\phi$, so just be a little careful.

• Motion in terms of Euler angles

+ For a rotating object, ϕ, θ, ψ change in time. The object (or equivalently the principal axes) has angular velocity

$$\vec{\omega} = \dot{\phi}\hat{k} + \dot{\theta}\hat{x} + \dot{\psi}\hat{e}_3$$

+ This is not written wrt a single set of orthogonal axes. With a little geometry, we can work out

$$\hat{k} = \cos\theta\, \hat{e}_3 + \sin\theta\,(\cos\psi\, \hat{e}_2 + \sin\psi\, \hat{e}_1)$$
$$\hat{x} = \cos\psi\, \hat{e}_1 - \sin\psi\, \hat{e}_2$$

(See rotation matrices in Cline or other texts)

$\Rightarrow$

$$\vec{\omega} = (\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\hat{e}_1 + (\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)\hat{e}_2 + (\dot{\psi} + \dot{\phi}\cos\theta)\hat{e}_3$$
$$\equiv \omega_1\hat{e}_1 + \omega_2\hat{e}_2 + \omega_3\hat{e}_3$$

+ The kinetic energy of rotation is

$$T = \tfrac{1}{2}I_1\omega_1^2 + \tfrac{1}{2}I_2\omega_2^2 + \tfrac{1}{2}I_3\omega_3^2$$

For a <u>symmetric body</u> $I_1 = I_2 \equiv I$, we can simplify to

$$T = \tfrac{1}{2}I(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \tfrac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2$$

+ Ex A symmetric top w/no torque has ω_3 = constant b/c ψ = cyclic. We recall that ω_1 + ω_2 oscillate, so $\vec{\omega}$ precesses around $\hat{e}_3$. Assuming θ = constant, the precession rate is $\dot{\psi}$. But $\hat{e}_3$ also wobbles around the conserved angular momentum (chosen along $\hat{k}$). The wobble rate is $\dot{\phi}$ (by definition of the Euler angle).

- Lagrangian Approach to Euler Equations

• ~~Lagrangian kinetic energy~~

+ The kinetic energy for translational (CM) and rotational motion is

$$T = \tfrac{1}{2} M \dot{\vec{x}}^2 + \tfrac{1}{2} I_1 \omega_1^2 + \tfrac{1}{2} I_2 \omega_2^2 + \tfrac{1}{2} I_3 \omega_3^2$$

+ The potential energy function can depend on position + orientation $V(\vec{x}, \phi, \theta, \psi)$

$$L = T - V$$

• The E-L eqn for ψ is

+ $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) = I_3 \dot{\omega}_3$$

+ $$\frac{\partial L}{\partial \psi} = I_1 \omega_1 \frac{\partial \omega_1}{\partial \psi} + I_2 \omega_2 \frac{\partial \omega_2}{\partial \psi} - \frac{\partial V}{\partial \psi}$$

but we can see

+ $$\partial \omega_1 / \partial \psi = \omega_2 \ , \quad \partial \omega_2 / \partial \psi = -\omega_1$$

+ Therefore, $$\frac{d}{dt}\left(\partial L / \partial \dot{\psi}\right) - \partial L / \partial \psi = 0$$

$\Rightarrow$

$$I_3 \dot{\omega}_3 + (I_2 - I_1)\omega_1 \omega_2 = -\partial V / \partial \psi$$

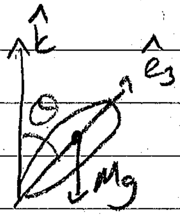
+ Assuming the generalized force $-\partial V/\partial \psi$ is the torque around $\hat{e}_3$, this is the Euler eqn for $\hat{e}_3$

• We could have chosen any principal axis to be $\hat{e}_3$, so the other Euler eqns are from $1 \to 2 \to 3 \to 1$ twice.

- Symmetric Top Supported By a Fixed Point

  - Torques from gravity

    + Consider a symmetric top fixed at one end with gravity acting at the center of mass. Mass $= M$, CM position $R\hat{e}_3$



    + Gravity exerts a torque $\vec{\tau} = (R\hat{e}_3) \times (-Mg\hat{k}) = MgR\sin\theta\,\hat{x}$ which is along the line of nodes (always).
      It will cause $\hat{e}_3$ to precess around $\hat{k}$ (if the top is already spinning around $\hat{e}_3$)

    + The precessional angular velocity is $\dot{\phi}$ from the definition of Euler angles

  - Lagrangian analysis

    + The Lagrangian is

$$L = \tfrac{1}{2} I (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \tfrac{1}{2} I_3 (\dot{\psi} + \dot{\phi}\cos\theta)^2 - MgR\cos\theta$$

      b/c the height of the CM is $R\cos\theta$.

    + Both ψ and ϕ are cyclic, so there are 2 constants of motion

angular momenta

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3(\dot{\psi} + \dot{\phi}\cos\theta) = \text{const}$$

$$p_\phi = \partial L / \partial \dot{\phi} = I\dot{\phi}\sin^2\theta + I_3(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta = \text{const}$$

    + We can invert these to

$$\dot{\phi} = \frac{p_\phi - p_\psi\cos\theta}{I\sin^2\theta}, \quad \dot{\psi} = \frac{p_\psi}{I_3} - \frac{p_\phi - p_\psi\cos\theta}{I\sin^2\theta}\cos\theta$$

    + Th

+ The E-L eqn for θ is nontrivial!

$$I\ddot{\theta} - I\dot{\phi}^2 \sin\theta\cos\theta + p_\psi \dot{\phi}\sin\theta - MgR\sin\theta = 0$$

If $\theta \neq$ constant, $\dot{\phi} \neq$ constant, and this is difficult to solve

+ When can we have steady precession $\theta = $ const, $\dot{\phi} =$ const?
For $\ddot{\theta} = 0$, we have a quadratic eqn for $\dot{\phi}$ w/ sol'n

$$\dot{\phi} = \frac{p_\psi \pm (p_\psi^2 - 4IMgR\cos\theta)^{1/2}}{2I\cos\theta}$$

This is only possible when $p_\psi^2 = I_3^2\omega_3^2 \geq 4IMgR\cos\theta$

+ When the force is small compared to the spin $p_\psi^2 \gg IMgR$,
the 2 solutions come from expanding the square root

$$\dot{\phi} \approx p_\psi / I\cos\theta \quad \text{fast precession (rarely seen)}$$

or

$$\dot{\phi} \approx \frac{MgR}{p_\psi} \quad \text{slow precession (see PHYS-3202)}$$

- Hamiltonian analysis + Nutation
  + In general, as the symmetric top precesses, θ is not constant, so the top bobs up + down. This bobbing motion is called nutation

+ The Hamiltonian is

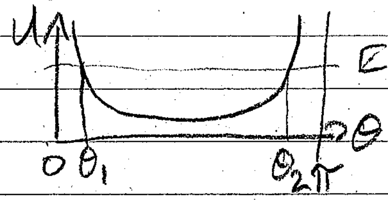
$$H = p_\theta\dot{\theta} + p_\psi\dot{\psi} + p_\phi\dot{\phi} - L = \ldots$$

$$= p_\theta^2/2I + p_\psi^2/2I_3 + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I\sin^2\theta} + MgR\cos\theta$$

+ We can write this in terms of an effective potential

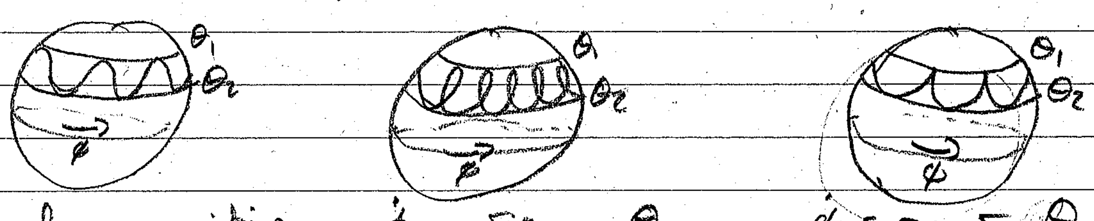
$$H = p_\theta^2/2I + U(\theta), \quad U(\theta) \equiv \frac{(p_\phi - p_\psi\cos\theta)^2}{2I\sin^2\theta} + MgR\cos\theta + \text{const}$$

+ Since the Hamiltonian $H = E$ = conserved energy, the motion of θ is (non-harmonic) oscillation between two values θ_1, θ_2 where $U(\theta_1) = U(\theta_2) = E$.



Steady precession is at the minimum of U
$E = U_{min}$

+ There are 3 cases of nutation (besides steady precession)



$\dot{\phi}$ always positive ($p_\phi > p_\psi \cos\theta_1$)

$\dot{\phi}$ negative near θ_1 ($p_\phi < p_\psi \cos\theta_1$)

$\dot{\phi} = 0$ at θ_1 ($p_\phi = p_\psi \cos\theta_1$)