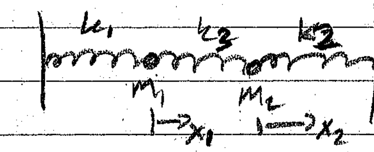
# Coupled Harmonic Oscillators

## ● General Problem + Approaches

- Example: 2 Masses on Springs

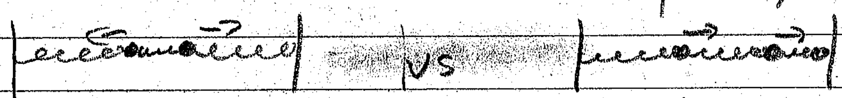



  • Suppose we have springs of constants $k_1$ + $k_2$ attached to 2 walls and 2 masses $m_1$ + $m_2$
    + The masses are connected by a 3rd spring of constant $k_3$
    + Measure displacement of each mass by $x_1, x_2$ from equilibrium position
    + For simplicity, let's take $m_1 = m_2 = m$, $k_1 = k_2 = k$, $k_3 = \bar{k}$.

  • Equations of motion
    + Excluding a constant, potential energy is
$$V(x_1, x_2) = \frac{1}{2} k(x_1^2 + x_2^2) + \frac{1}{2} \bar{k}(x_2 - x_1)^2$$
    so
$$L = \frac{1}{2} m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2} k(x_1^2 + x_2^2) - \frac{1}{2} \bar{k}(x_2 - x_1)^2$$
    + The EOM are
$$m\ddot{x}_1 + kx_1 + \bar{k}(x_1 - x_2) = 0$$
$$m\ddot{x}_2 + kx_2 + \bar{k}(x_2 - x_1) = 0$$
    * Coupled 2nd order ODEs. Different than circular!

  • Guess oscillating solution $x_1 = B_1 e^{i\omega t}$, $x_2 = B_2 e^{i\omega t}$
    + Then we find a matrix eqn
$$\begin{bmatrix} -m\omega^2 + (k + \bar{k}) & -\bar{k} \\ -\bar{k} & -m\omega^2 + (k + \bar{k}) \end{bmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = 0$$
    + Looks like an e'value problem. Characteristic eqn is
$$(k + \bar{k} - m\omega^2)^2 - \bar{k}^2 = 0 \Rightarrow \omega^2 = \frac{1}{m}(k + \bar{k} \pm \bar{k})$$
    So there are 2 sol'ns w/ frequencies
$$\omega_1 = \sqrt{(k + 2\bar{k})/m}, \quad \omega_2 = \sqrt{k/m}$$
    + Solution of freq $\omega_1$ has $B_1 = -B_2$ while freq $\omega_2$ sol'n has $B_1 = B_2$
    This means the higher freq $\omega_1$ sol'n is

antisymmetric motion while the lower freq is symmetric




The symmetric mode does not stretch the $\bar{k}$ spring

+ The general solution is

$$x_1 = A_1 e^{i(\omega_1 t+\theta_1)} + A_2 e^{i(\omega_2 (t+\theta_2)}, \quad x_2 = -A_1 e^{i(\omega_1 t+\theta_1)} + A_2 e^{i(\omega_2 t+\theta_2)}$$

or, taking the real part

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\omega_1 t+\theta_1) + A_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_2 t+\theta_2)$$

- Small Deviations from Equilibrium: Oscillators Everywhere
  - Consider a general Lagrangian
    $$L = T(q,\dot{q}) - V(q)$$
    + Suppose there is an equilibrium point where $q_i = q_i^{(0)}$. In other words, this is a time-indep. solution of EOM.
    + We want to understand motion close to the equilibrium. Take $q_i = q_i^{(0)} + \delta q_i(t)$ and expand the Lagrangian in powers of $\delta q$.
    + Kinetic energy is generally

$$T = \frac{1}{2}\sum_{i,j} T_{ij}(q)\,\dot{q}_i\dot{q}_j = \frac{1}{2}\sum_{i,j} T_{ij}(q^{(0)}+\delta q)\,\delta\dot{q}_i\,\delta\dot{q}_j$$

$$= \frac{1}{2}\sum_{i,j}\left(m_{ij}\,\delta\dot{q}_i\,\delta\dot{q}_j + \dots\right) \simeq \frac{1}{2}\sum_{i,j} m_{ij}\,\delta\dot{q}_i\,\delta\dot{q}_j$$

where

$$m_{ij} \equiv T_{ij}(q^{(0)}) = \text{constant } \underline{\text{mass}} \text{ matrix}$$

+ Meanwhile, potential energy is

$$V(q) = V(q^{(0)}) + \sum_i \frac{\partial V}{\partial q_i}(q^{(0)})\,\delta q_i + \frac{1}{2}\sum_{i,j}\frac{\partial^2 V}{\partial q_i \partial q_j}(q^{(0)})\,\delta q_i\,\delta q_j + \dots$$

The 1st term is constant (irrelevant).
The 2nd term vanishes b/c force must $=0$ at equilibrium
So

$$V(q) \simeq \frac{1}{2}\sum_{i,j} V_{ij}\,\delta q_i\,\delta q_j \quad \text{w/ } V_{ij} = \text{Hessian of 2}^{\text{nd}} \text{ derivatives}$$

+ To 2nd order, $L = \frac{1}{2}\sum_{i,j}\left(m_{ij}\,\delta\dot q_i\,\delta\dot q_j - V_{ij}\,\delta q_i\,\delta q_j\right)$

From now on we'll rename $\delta q \to q$ for simplicity

+ EOM are

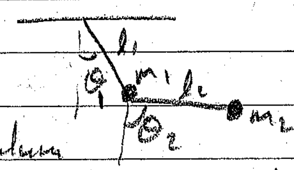
$$\sum_j m_{ij}\ddot q_j + \sum_j V_{ij} q_j = 0$$

If you choose linear combinations correctly, you get a form

$$\ddot q_1 + k_{11} q_1 + k_{12} q_2 + \dots = 0$$

This is the same as the coupled spring example.

· Example: Double Pendulum




+ Consider a single pendulum of mass $m_2$, length $l_2$ hanging from a simple pendulum of mass $m_1$, length $l_1$. Each has angle $\theta_1, \theta_2$ from vertical

+ The positions are

$$x_1 = l_1 \sin\theta_1\,,\quad y_1 = -l_1\cos\theta_1 \quad \text{(relative to top support)}$$

$$x_2 = x_1 + l_2 \sin\theta_2\,,\quad y_2 = y_1 - l_2\cos\theta_2$$

So

$$L = \tfrac{1}{2}m_1 l_1^2\dot\theta_1^2 + \tfrac{1}{2}m_2\left[l_1^2\dot\theta_1^2 + l_2^2\dot\theta_2^2 + 2l_1 l_2\cos(\theta_1-\theta_2)\dot\theta_1\dot\theta_2\right] + m_1 g l_1\cos\theta_1 + m_2 g l_1\cos\theta_1 + m_2 g l_2\cos\theta_2$$

+ Expanding out,

$$L = \tfrac{1}{2}m_1 l_1^2\dot\theta_1^2 + \tfrac{1}{2}m_2\left(l_1\dot\theta_1 + l_2\dot\theta_2\right)^2 - \tfrac{1}{2}(m_1 l_1 + m_2 l_1)g\theta_1^2 - \tfrac{1}{2}m_2 l_2 g\theta_2^2$$

+ EOM are

$$(m_1+m_2)l_1^2\ddot\theta_1 + m_2 l_1 l_2\ddot\theta_2 + (m_1 l_1 + m_2 l_1)g\theta_1 = 0$$

$$m_2 l_1 l_2\ddot\theta_1 + m_2 l_2^2\ddot\theta_2 + m_2 l_2 g\theta_2 = 0$$

## – Normal Modes: Solving at level of EOM

• We expect oscillatory solutions. + We look for solutions where all coordinates $q_i$ oscillate w/ the same frequency, or $q_i = B_i e^{i\omega t}$ for complex constants $B_i$

+ These are normal modes.

+ A general solution is a superposition of normal modes b/c the EOM are linear

- Generalized eigenvalue problem
  + Write our guessed solution as

$$\begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{bmatrix} e^{i\omega t}$$

  + The EOM becomes the matrix eqn

$$-\omega^2 \begin{bmatrix} M_{ij} \end{bmatrix} \begin{bmatrix} B_1 \\ \vdots \\ B_N \end{bmatrix} = -\begin{bmatrix} V_{ij} \end{bmatrix} \begin{bmatrix} B_1 \\ \vdots \\ B_N \end{bmatrix}$$

This is a generalized version of the e'value/e'vector problem
  + To find eigenvalues/normal mode frequencies, note

$$\det \left( V - m\omega^2 \right) = 0 \quad \text{with } V, m = \text{matrices}$$

Since V and m are real symmetric, each e'value $\omega^2$ is real.
  + Oscillatory normal modes have $\omega_{(n)}^2 > 0$;
    If $\omega^2 < 0$, the solution is a real exponential, meaning the equilibrium is unstable (rolling off a hill)
  + Given an (real, oscillatory) frequency $\omega_{(n)}$, the normal modes are given by solving the vector eqn

$$\omega_{(n)}^2 M \vec{B}_{(n)} = V \vec{B}_{(n)}$$

  + If $B_{i(n)} = A_{i(n)} e^{i\theta_{i(n)}}$, general (real) solution is

$$q_i(t) = \sum_n A_{i(n)} \cos(\omega_{(n)} t + \theta_{i(n)})$$

  in terms of the normal modes

- Normal Coordinates
  + We can show that the normal modes are orthogonal w/ generalized inner product

$$\vec{B}_{(n)}^T M \vec{B}_{(p)} = 0 \quad n \neq p$$

  This follows b/c

$$\omega_{(p)}^2 \vec{B}_{(n)}^T m \vec{B}_{(p)} = \vec{B}_{(n)}^T V \vec{B}_{(p)} = \omega_n^2 \vec{B}_{(n)}^T m \vec{B}_{(p)}$$

  So it vanishes for different frequencies.
  (With degeneracy, $\omega_{(p)} = \omega_{(n)}$, we can use Gram-Schmidt)

  + We can also normalize to $\vec{B}_{(n)}^T m \vec{B}_{(p)} = \delta_{np}$.
  Then we can write coords w/ an extra amplitude

$$q_i = \sum_n \alpha_{(n)} B_{i(n)} e^{i\omega_n t} \equiv q_i = \sum_n B_{i(n)} \eta_n(t)$$

The $\eta_n(t)$ variables are normal coordinates. They oscillate with only one frequency

- Normal Coordinates Directly (in Lagrangian)

Let's work w/o using EOM.

- We can start with kinetic energy $T = \frac{1}{2}\sum_{ij} m_{ij}\dot{q}_i\dot{q}_j$
  - Want to write it as the form
    $$T = \tfrac{1}{2}M_1\dot{\tilde{q}}_1\dot{\tilde{q}}_1 + \tfrac{1}{2}M_2\dot{\tilde{q}}_2\dot{\tilde{q}}_2 + \dots$$
  - Sometimes we can work out the relation of $q \to \tilde{q}$ by inspection or term by term
  - More generally, we want to diagonalize $[m_{ij}]$, which will have eigenvalues $M_a$
  - Normalize the eigenvectors $\tilde{B}_{(n)}$ usually as $\tilde{B}^T_{(\alpha)}\tilde{B}_{(\beta)} = \delta_{\alpha\beta}$

    Then $\tilde{q}_n = \tilde{B}^T_{i(a)} q_i$
  - Finally, define $\tilde{\eta}_a = \sqrt{M_a}\,\tilde{q}_a$, so
    $$T = \tfrac{1}{2}\sum_a \dot{\tilde{\eta}}_a^2$$

- Next, redefine the potential in terms of $\tilde{\eta}$ variables
  - We get $V = \frac{1}{2}\sum_{ij} V_{ij} q_i q_j = \frac{1}{2}\sum_{ab} k_{ab}\tilde{\eta}_a\tilde{\eta}_b$
  - You may be able to do this already in expanding the full potential
  - Now diagonalize $[k_{ab}]$. Eigenvalues are $\omega_{(n)}^2$ and eigenvectors $\vec{B}_{a(n)}$. Normalize s.t. $\vec{B}^T_{(n)}\vec{B}_{(p)} = \delta_{np}$
  - Finally define normal coordinates
    $$\eta_n = B_{a(n)}\tilde{\eta}_a,$$
    so $$V = \tfrac{1}{2}\sum_n \omega_{(n)}^2\eta_n^2$$

- Since $\vec{B}_{(n)}$ are orthonormal, the form of the kinetic energy is unchanged
  $$L = \tfrac{1}{2}\sum_n \dot{\eta}_n^2 - \tfrac{1}{2}\sum_n \omega_{(n)}^2\eta_n^2$$

The $\eta_n$ are the normal coordinates

- Examples
  - Revisit our 2 masses on 3 springs
    + We found a solution

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i\omega_1 t + \theta_1} + A_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i\omega_2 t + \theta_2}$$

    + With normalization, these evectors give normal mode eigenvectors

$$B_{(1)} = \frac{1}{\sqrt{2m}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad B_{(2)} = \frac{1}{\sqrt{2m}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

    + The normal coordinates are therefore

$$x_1 = \frac{1}{\sqrt{2m}}(\eta_1 + \eta_2), \quad x_2 = \frac{1}{\sqrt{2m}}(\eta_2 - \eta_1)$$

$$\eta_1 = \sqrt{\frac{m}{2}}(x_1 - x_2), \quad \eta_2 = \sqrt{\frac{m}{2}}(x_1 + x_2)$$

    + We can see that

$$L = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2} k (x_1^2 + x_2^2) - \frac{1}{2} k_c (x_2 - x_1)^2$$

$$= \frac{1}{2} \dot{\eta}_1^2 + \frac{1}{2} \dot{\eta}_2^2 - \frac{1}{2} \left(\frac{k + 2k_c}{m}\right) \eta_1^2 - \frac{1}{2} \frac{k}{m} \eta_2^2$$

    + This follows the 1st approach & confirms the normal mode frequencies in the Lagrangian

  - Double pendulum again
    + In terms of the angles,

$$L = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 (l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2)^2 - \frac{1}{2}(m_1 l_1 + m_2 l_1) g \theta_1^2 - \frac{1}{2} m_2 l_2 g \theta_2^2$$

    + Now we can define

$$\tilde{\eta}_1 = \sqrt{m_1}\, l_1 \theta_1, \quad \tilde{\eta}_2 = \sqrt{m_2}(l_1 \theta_1 + l_2 \theta_2)$$

so

$$L = \frac{1}{2} \dot{\tilde{\eta}}_1^2 + \frac{1}{2} \dot{\tilde{\eta}}_2^2 - \frac{1}{2} \frac{m_1 + m_2}{m_1 l_1} g \tilde{\eta}_1^2 - \frac{1}{2} \frac{g}{l_2} \left(\tilde{\eta}_2 - \sqrt{\frac{m_2}{m_1}} \tilde{\eta}_1\right)^2$$

    + The potential matrix is

$$[k] = g \begin{bmatrix} \frac{m_1 + m_2}{m_1 l_1} + \frac{m_2}{m_1 l_2} & -\sqrt{\frac{m_2}{m_1}} \frac{1}{l_2} \\ -\sqrt{\frac{m_2}{m_1}} \frac{1}{l_2} & 1/l_2 \end{bmatrix}$$

+ For $m_1 \gg m_2$, the frequencies + (unnormalized) normal modes are approx

$$\omega_1^2 \approx g/l_1, \quad \begin{bmatrix}\hat{\eta}_1\\ \hat{\eta}_2\end{bmatrix} \approx \begin{bmatrix}1\\ \sqrt{\frac{m_2}{m_1}}\frac{l_1}{l_1-l_2}\end{bmatrix} \Rightarrow \frac{\theta_1}{\theta_2} \approx \frac{l_1-l_2}{l_1} \approx \mathcal{O}(1)$$

and

$$\omega_2^2 \approx g/l_2, \quad \begin{bmatrix}\hat{\eta}_1\\ \hat{\eta}_2\end{bmatrix} \approx \begin{bmatrix}\sqrt{\frac{m_2}{m_1}}\frac{l_1}{l_2-l_1}\\ 1\end{bmatrix} \Rightarrow \frac{\theta_1}{\theta_2} \approx \frac{m_2}{m_1}\frac{l_2}{l_2-l_1} \ll 1$$

In the 2nd, this is almost the lower pendulum swinging with the upper stationary. (May see other extreme on HW)

- Weak Coupling + Forcing
  - ~~Weak Coupling~~ is the case that we have basically 2 separate oscillators that affect each other just a little.

- Let's go back to our 3 spring prototype example, first with all springs different but $m_1 = m_2 = m$
  + We chose
  $$L = \tfrac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \tfrac{1}{2}k_1 x_1^2 - \tfrac{1}{2}k_2 x_2^2 - \tfrac{1}{2}\bar{k}(x_2 - x_1)^2$$
  or
  $$[V_{ij}] = \begin{bmatrix} k_1 + \bar{k} & -\bar{k} \\ -\bar{k} & k_2 + \bar{k}\end{bmatrix} \quad \text{with } \bar{k} \ll k_1, k_2$$
  + The characteristic eqn is
  $$\det\begin{bmatrix} k_1+\bar{k}-m\omega^2 & -\bar{k} \\ -\bar{k} & k_2+\bar{k}-m\omega^2\end{bmatrix} \approx (m\omega^2 - k_1)(m\omega^2 - k_2) + \dots = 0$$
  The frequencies are $\omega_{(1)} \approx \sqrt{k_1/m}$, $\omega_{(2)} \approx \sqrt{k_2/m}$
  + In each case, (unnormalized) normal mode vectors are
  $$\approx \begin{bmatrix}1\\ \bar{k}/(k_2-k_1)\end{bmatrix} \quad \text{and} \quad \approx \begin{bmatrix}\bar{k}/(k_1-k_2)\\ 1\end{bmatrix}.$$
  + These are almost just motion of $x_1$ & $x_2$
  + Suppose we have initial conditions $x_1 = a$, $x_2 = 0$, $\dot{x}_1 = \dot{x}_2 = 0$. Then
  $$x_1 = a\cos(\omega_1 t) + \dots$$
  $$x_2 = a\bar{k}/(k_2-k_1)\left[\cos(\omega_1 t) - \cos(\omega_2 t)\right] + \dots$$
  This is large amplitude motion of $x_1$ + small of $x_2$

- But take this example with degeneracy $k_1 = k_2 = k$.
  + From before we know
  $$\omega_{(1)}^2 = \frac{k+2\bar{k}}{m}, \quad \omega_{(2)}^2 = k/m$$

with normal modes $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

+ Even if $\bar{k} \ll k$, these normal modes are very different from $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. (This applies if $k_1 \neq k_2$ but $\bar{k} \gg k_1 - k_2$ also)

+ If we take those same initial conditions

$$x_1 = \frac{a}{2}\left[\cos(\omega_1 t) + \cos(\omega_2 t)\right]$$
$$x_2 = a/2\left[\cos(\omega_2 t) - \cos(\omega_1 t)\right]$$

With trig identities, these are

$$x_1 = a\cos\left[(\omega_1+\omega_2)t/2\right]\cos\left[(\omega_1-\omega_2)t/2\right]$$
$$x_2 = a\sin\left[(\omega_1+\omega_2)t/2\right]\sin\left[(\omega_1-\omega_2)t/2\right]$$

+ Now we expand in $\bar{k}/k = \epsilon$.

$$\omega_1+\omega_2 \simeq 2\sqrt{k/m}\ , \quad \omega_1-\omega_2 \simeq \epsilon\sqrt{k/m}$$

Therefore

$$x_1 = a\cos(\epsilon\omega_2 t/2)\cos(\omega_2 t)\ , \quad x_2 = a\sin(\epsilon\omega_2 t/2)\sin(\omega_2 t)$$

where $\omega_2$ = freq of the uncoupled oscillator.

+ These are oscillations $90°$ out of phase with freq = single oscillator freq, and an amplitude envelope with a frequency = beat frequency.

- Forcing may apply at weak or strong coupling
  + It's usually easiest to write this in terms of normal coordinates b/c they have definite frequencies
  + So take our degenerate 3-spring example. Recall

$$\eta_1 = \sqrt{m/2}\,(x_1 - x_2)\ , \quad \eta_2 = \sqrt{m/2}\,(x_1 + x_2)$$

  + If a force acts on the $1^{st}$ mass, the EOM are

$$m\ddot{x}_1 + \cdots = F(t)\ , \quad m\ddot{x}_2 + \cdots = 0$$
$$\Rightarrow \quad \ddot{\eta}_1 + \omega_1^2\eta_1 = F(t)/\sqrt{2m}\ , \quad \ddot{\eta}_2 + \omega_2^2\eta_2 = F(t)/\sqrt{2m}$$

Solve each as usual, add back to get $x_1, x_2$

  + Of course you can change forcing, apply to nondegenerate oscillators, etc.