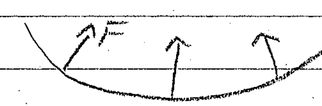
More on the Gravitational Potential

- **Use of Potentials vs Forces / General Comments**
  - We can think of the potential energy on an object of mass m as $V(\vec{r}) = m\Phi(\vec{r})$, so $\vec{F} = -m\vec{\nabla}\Phi$
    - Same idea as electrostatic case
    - Then
    $$\Phi(\vec{r}) = -\int d^3r' \, \frac{G\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$
    is the integral over the mass distribution that acts on m
    - You can also add a potential corresponding to the centrifugal force

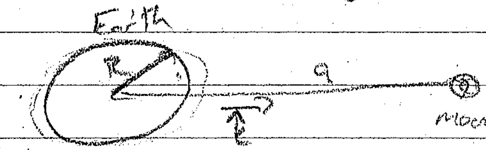
  - Potential determines shape of objects held together by gravity
    - Consider a fluid surface.



      + The force on fluid is $\perp$ to surface in equilibrium, or else fluid would flow
      + This means $\vec{a}\cdot\vec{\nabla}\Phi = 0$ for any tangent vector to the surface $\Rightarrow$ surface is an equipotential surface (has constant potential)

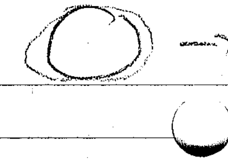
    - Over long times, even the earth is effectively fluid, so the planet's surface is equipotential

- **Tides:**
  - These are a daily variation in height of ocean's due to moon's (and, not quite as much, the sun's) gravity



    - In shorthand, the force on earth due to moon differs on opposite sides of the earth

    - The "tidal force" is
    $$\Delta\vec{F} = \vec{F}_{near} - \vec{F}_{far} = \frac{GM_\oplus M_m}{(a-R)^2}\hat{r} - \frac{GM_\oplus M_m}{(a+R)^2}\hat{r} \simeq \frac{4GM_\oplus M_m R}{a^3}\hat{r}$$



• The near oceans are pulled toward the moon more than the rocky earth, which is pulled more toward moon than far oceans.

$\Rightarrow$ 2 high tides per day

– Potential Treatment

• Treat the moon as a point object at position $\vec{a} = a\hat{z}$ from the center of earth

+ The potential at any point $\vec{r}$ on earth's surface is

$$\Phi(\vec{r}) = \frac{-G \cdot M_m}{|\vec{r} - \vec{a}|} = \frac{-G M_m}{[r^2 + a^2 - 2ar\cos\theta]^{1/2}}$$

+ Since $R_\oplus \ll a$, we expand

$$\Phi \simeq \frac{-GM_m}{a}\left[1 + \frac{r}{a}\cos\theta - \frac{1}{2}\left(\frac{r}{a}\right)^2 + \frac{3}{2}\left(\frac{r}{a}\cos\theta\right)^2 + \dots\right]$$

from $(1+x)^{-1/2} \simeq 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \dots$
+ keeping only linear term for $x = (r/a)^2$.

+ First term is a constant. 2nd term
$\Phi \simeq -GM_m z/a^2 \Rightarrow$ uniform pull toward moon
The last varies with position over the earth, leads to force partly inward along x, y plane (tidal force) + toward/away from moon

• Importance of the tidal forces:

+ First, work in a frame of earth accelerating toward moon to remove that force

+ Then, compared to gravitational field of earth,

$$\frac{\nabla\Phi}{\left(\frac{GM_\oplus}{R_\oplus^2}\right)} \simeq \frac{M_m R_\oplus/a^3}{M_\oplus/R_\oplus^2} = \frac{M_m R_\oplus^3}{M_\oplus a^3} \sim 5.7\times10^{-8}$$

+ Tidal force from sun is about $\frac{1}{2}$ as big ← coincidence
+ This is much smaller than various effects on earth, like centrifugal force. But they are noticeable b/c they change over the course of the day & month, ie, 2 high tides per day of different sizes over the month.

- Estimated Heights of tides
  + Strict assumptions: earth is perfectly rigid sphere covered in water, which is always in equilibrium
  + Including the gravitational potential from the earth, the water experiences potential (ignoring centrifugal effects)

$$\Phi = \left[g h(\theta) - \frac{GM_\oplus}{R_\oplus}\right] - \frac{GM_m}{a^3}\left(R_\oplus + h(\theta)\right)^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

where $h(\theta)$ = height at θ

  + The surface of the water should be at constant potential. For $h(\theta) \ll R_\oplus$

$$h(\theta) \approx \frac{GM_m R_\oplus^2}{g a^3}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) = \frac{R_\oplus M_m}{M_\oplus}\left(\frac{R_\oplus}{a}\right)^3\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

  + The scale is $\frac{M_m}{M_\oplus}\left(\frac{R_\oplus}{a}\right)^3 R_\oplus \approx 0.36\,m$ for the moon. This seems small, but is reasonable for mid-ocean. Tides are greatly affected by the continental shelves and also possible resonances in places.

- Shapes of potentials for general matter distributions
  What is the potential like when mass is not spherically symmetric?

  — The potential far away: answer usually complicated, but simplifies & has universal features at long distance

  - The potential from a general distribution of mass at point $\vec{r}$ is

$$\Phi(\vec{r}) = -G\int d^3\vec{r}\,'\,\frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} = -G\int d^3\vec{r}\,'\,\frac{\rho(\vec{r}\,')}{(r^2+r'^2-2\vec{r}\cdot\vec{r}\,')^{1/2}}$$

+ Suppose $\rho(\vec{r}\,') = 0$ for $|\vec{r}\,'| < R$ (some fixed distance)
Then consider only points $|\vec{r}| \gg R$.

+ We can expand

$$\Phi(\vec{r}) = -G \int d^3\vec{r}\,' \frac{\rho(\vec{r}\,')}{r} \left[1 - \frac{2\vec{r}\cdot\vec{r}\,'}{r^2} + \frac{r'^2}{r^2} \right]^{-1/2}$$

$$\equiv \Phi_0(\vec{r}) + \Phi_1(\vec{r}) + \Phi_2(\vec{r}) + \ldots$$

- The contributions:
  + $\Phi_0(\vec{r}) = -GM/r$, where M = total mass

  + $\Phi_1(\vec{r}) = -2\frac{G}{r^3} \left(\int d^3\vec{r}\,' \rho(\vec{r}\,') \vec{r}\,' \right) \cdot \vec{r} = -\frac{2GM}{r^3} \vec{r} \cdot \vec{R}$
  where $\vec{R}$ = CM position. We can always make this term vanish (for gravity) by shifting origins

  + In EM, Φ_1 is the dipole term that can be nonvanishing if the total charge is zero.

  + $\Phi_2(\vec{r}) = -\frac{G}{r^5} \int d^3\vec{r}\,' \rho(\vec{r}\,') \left[\frac{3}{2}(\vec{r}\cdot\vec{r}\,')^2 - \frac{1}{2}(r')^2 r^2 \right]$

$$= -\frac{G}{2r^5} \vec{r} \cdot (\overleftrightarrow{Q}\vec{r})$$

for quadrupole moment tensor

$$Q_{ij} = \int dm' \left[3 r_i' r_j' - (r')^2 \delta_{ij} \right]$$

(This is clearly related to the inertia tensor.)

+ If $\rho(\vec{r}\,')$ is axially symmetric around the z axis,

$$Q_{xy} = Q_{xz} = Q_{yz} = 0, \quad Q_{zz} = -2Q_{xx} = -2Q_{yy} \equiv Q$$

where

$$Q = \int dm' \left(2z'^2 - x'^2 - y'^2 \right)$$

so

$$\Phi_2(\vec{r}) = -\frac{GQ}{2r^3} (3\cos^2\theta - 1)$$ for polar angle θ.

- Effects on + of Earth

  - The earth is oblate (bulged out in middle) [vs prolate] due to centrifugal force.

    + Suppose Earth's equatorial radius is $R_\oplus$ and polar radius is $R_\oplus(1-\epsilon)$. By measurement $\epsilon \approx 1/300$. Let's understand why.

    + The centrifugal force of earth's rotation is $\frac{1}{2} m\omega^2 \vec{\rho}$ on a mass m, so $\Phi_c = -\frac{1}{2}\omega^2 r^2 \sin^2\Theta$. (cylindrical) The earth's surface, over geologic time, should become an equipotential, but $\Phi_0 + \Phi_c \neq$ constant. There must also be a quadrupole

    + Treat the oblate spheroid shape of the Earth as an ellipse of revolution with $\epsilon \ll 1$. Then

$$\frac{r^2\sin^2\Theta}{R_\oplus^2} + \frac{r^2\cos^2\Theta}{R_\oplus^2(1-\epsilon)} = 1 \Rightarrow r \approx R_\oplus(1-\epsilon\cos^2\Theta)$$

    Further the quadrupole is

$$Q = -\tfrac{4}{5} M R_\oplus^2 \epsilon \quad \text{assuming uniform density}$$

    + Now we have $\Phi = \Phi_0 + \Phi_2 + \Phi_c = \text{const}$, so the $\cos^2\Theta$ terms must cancel. We get

$$-\frac{GM}{R_\oplus}\epsilon + \frac{3}{5}\frac{GMR_\oplus^2}{R_\oplus^3}\epsilon + \frac{1}{2}\omega^2 R_\oplus^2 = 0$$

$$\Rightarrow \epsilon \approx 5\omega^2 R_\oplus^2/4g$$

    where $g = GM/R_\oplus^2$ is gravitational acceleration. This number is a little too large b/c earth's core is more dense

  - Some effects
    + The quadrupole force is not central, so the orbits of satellites precess around the earth's rotational axis.

+ The oblateness of the sun also causes the orbits of the planets to precess slightly. For Mercury, there is also a (measurable) but ~10x smaller effect from GR.