PHYS-3202 Homework 4 Due 2 Oct 2019

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Spherical Polar Unit Vectors**

    In this problem, you will find the components of the spherical polar unit vectors $\hat{r}, \hat{\theta}, \hat{\phi}$ in terms of the Cartesian unit vectors $\hat{i}, \hat{j}, \hat{k}$.

    (a) We know that the velocity of an object is given by $\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$. Begin by finding expressions for $\dot{x}$, $\dot{y}$, and $\dot{z}$ in terms of spherical polar coordinates r, θ, ϕ and their time derivatives.

    (b) By comparing to the form of $\vec{v}$ in spherical coordinates, find $\hat{r}, \hat{\theta}, \hat{\phi}$.

    (c) Use your results from the previous two parts to find the acceleration vector in spherical polar components (that is, as components multiplying $\hat{r}, \hat{\theta}, \hat{\phi}$). *Hint:* $\hat{i}, \hat{j}, \hat{k}$ are time-independent.

2. **Projectile Range with Linear Air Resistance** *inspired by KB 3.9*

    In this problem, consider projectile motion with linear air resistance given by coefficient $\gamma = \lambda/m$ as discussed in the class notes. Assume that the projectile has initial postion $\vec{r} = 0$ and initial velocity $\vec{v}_0 = u\hat{i} + w\hat{k}$ with $w > 0$.

    (a) At what time does the projectile reach its maximum height? What is the maximum height that the particle obtains?

    (b) Is the time it takes to fall back to $z = 0$ equal to, less than, or greater than the time to reach the maximum height? Give a qualitative explanation. *Hint:* Find the height of the projectile after it has been falling for the same length of time as it rose. Then argue that this position is always positive by noting that it is zero when $w = 0$ and has non-negative derivative with respect to w.

    (c) Argue that the total time for the projectile's flight (from $z = 0$ at $t = 0$ to $z = 0$ again) is approximately $t \approx (w/g + 1/\gamma)(1 - e^{-1-\gamma w/g})$ for large γ by using an iterative solution (find t first for $\gamma \to \infty$, then use that value in the exponential). Using this value of time, find the range (ie, total horizontal distance traveled).

3. **An Extended Isotropic Spring** *from KB 3.11 and 4.10*

    One end of a spring is attached to the origin, and the other is attached to a mass m. The spring can swivel freely in two dimensions (horizontal, so we neglect gravity). The equilibrium length of the spring is a, so the restoring force on the mass is $-k(\rho - a)$ in cylindrical coordinates. Ignore damping. Note that cylindrical coordinates in the xy plane are the same as spherical polar coordinates in the equatorial plane with some renaming.

    (a) Suppose the mass oscillates linearly (ie, in the x direction with $y = 0$ at all times). What is the angular frequency ω_0 of oscillation?

    (b) For motion with angular momentum $\vec{J} = J\hat{z}$, find the effective potential for radial motion.

    (c) Suppose the mass has initial conditions $\rho = a, \varphi = 0$ and $\dot{\rho} = 0, \dot{\varphi} = \omega$. If the mass reaches a maximum radius of $2a$, find ω and the value of $\dot{\varphi}$ when $\rho = 2a$. Write your answers as multiples of the natural frequency ω_0.

(d) Suppose instead that the initial conditions are $\rho = a, \varphi = 0$ and $\dot{\rho} = a\omega, \dot{\varphi} = \omega$. The mass still reaches a maximum radius of $2a$. Find ω in this case as a multiple of ω_0.

(e) Now suppose that the mass is moving in a circular orbit of radius $\rho = 2a$. Find the angular velocity of the orbit (which is also the frequency of the orbit) in terms of the natural oscillator frequency ω_0. *Hint:* Where is the circular orbit in terms of the effective potential for radial motion?

(f) The mass is in the circular orbit described in part (e). At time $t = 0$, it experiences a sharp radially directed force, so it gains a small component of velocity in the $\hat{\rho}$ direction, which leaves the angular momentum unchanged. The mass subsequently oscillates around the circular orbit. Find the frequency ω' of this oscillation in terms of ω_0 and give a qualitative description of the mass's motion (you will need the answer from part (e) also). *Hint:* Think about the expansion of the effective potential around the circular orbit.