PHYS-3202 Homework 3 Due 25 Sept 2019

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Work Done on a Forced Oscillator** *from KB 2.25*

   Consider a harmonic oscillator with damping γ and natural frequency ω_0 that experiences a force $F(t) = F\cos(\omega t)$. In the following, use the real solution for the oscillator position and neglect transients.

   (a) Recall that the power, or work done on an object per unit time, is $P = F\dot{x}$. Find the power on the oscillator by the force $F(t)$ at time t. Then find its average over one period.

   (b) The damping force is $-2m\gamma\dot{x}$. Find the power by the damping force at time t and the average power over one period. Show that your result is the opposite of the power from the driving force, so the total work done on the oscillator over a period is zero.

2. **Damped Unstable Equilibrium**

   Consider movement around the top of a parabolic potential including damping and an external driving force. The equation of motion is

   $$\ddot{x} + 2\gamma\dot{x} - \kappa^2 x = F(t)/m \ , \tag{1}$$

   where γ and κ are positive constants.

   In the first 2 parts, set the driving force $F(t) = 0$.

   (a) Find the general solution to (1). Show that x grows exponentially in t after a short period of time.

   (b) Assume x is given by the exponentially growing solution only. Show that the acceleration term in (1) is negligible when $\kappa \ll \gamma$. In other words, show that $|\ddot{x}| \ll 2\gamma|\dot{x}|$ and $|\ddot{x}| \ll \kappa^2|x|$. Physics very similar to this is important in the theory of inflation, which postulates that the early universe expanded very rapidly.

   (c) Now consider sinusoidal forcing $F(t) = Fe^{i\omega t}$. Write the general solution, including the solutions to the non-driven equation. Is it possible to neglect these extra terms after sufficient time passes?

3. **Half-Wave Forcing**

   Consider a damped harmonic oscillator subject to the driving force

   $$F(t) = \begin{cases} F\sin(\omega t) & (0 < t < \pi/\omega) \\ 0 & (\pi/\omega < t < 2\pi/\omega) \end{cases} \ . \tag{2}$$

   In other words, the driving force is the positive part of a sine wave. The oscillator has natural frequency ω_0 and quality factor Q.

   (a) Write $F(t)$ as a complex Fourier series.

   (b) Write the solution for the oscillator as a Fourier series, neglecting transients.

   (c) Suppose that $\omega_0 = 2\omega$ and the quality factor $Q \gg 1$. Argue that oscillator position $x(t)$ is dominated by a single frequency, and write $x(t)$ in that approximation. Make sure to simplify the amplitude and phase lag as much as possible.

4. **A Couple of Vector Identities**

(a) *from KB A.7* Using vector triple-product identities, write $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$ in terms of the dot products $\vec{a} \cdot \vec{c}$, $\vec{b} \cdot \vec{c}$, $\vec{a} \cdot \vec{d}$, and $\vec{b} \cdot \vec{d}$

(b) *from KB A.8* Verify the identity $\vec{\nabla} \times (\vec{a} \times \vec{b}) = (\vec{\nabla} \cdot \vec{b})\vec{a} + (\vec{b} \cdot \vec{\nabla})\vec{a} - (\vec{\nabla} \cdot \vec{a})\vec{b} - (\vec{a} \cdot \vec{\nabla})\vec{b}$ by comparing the z component of each side of the identity.