PHYS-3202 Homework 2 Due 18 Sept 2019

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Bouncing Ball**

   A ball is released from rest at height h and bounces off the floor with coefficient of restitution e for each bounce. Treat its motion as entirely one-dimensional.

   (a) *KB 2.28* Show that the ball comes to rest on the floor at time

   $$t = \frac{1+e}{1-e}\sqrt{\frac{2h}{g}} \tag{1}$$

   (including the time before the first bounce).

   (b) Find the total distance that the ball travels including the distance before the first bounce.

2. **Falling Water** *inspired by FC 2.20*

   A spherical drop of water of radius r_0 and uniform density ρ_0 nucleates inside a cloud and falls due to gravity, gathering mass from water vapor in the cloud and growing. As it falls for a time dt, it collects the mass $dm = \rho_1 \pi r^2 v dt$ of the cylinder it sweeps out in the cloud, where r and v are its radius and velocity as a function of time and ρ_1 is the density of water vapor in the cloud. Assume that the water drop keeps the same density and that $\rho_1 \ll \rho_0$.

   (a) Show that $\dot{r} = \rho_1 v/4\rho_0$ and find the equation of motion in terms of r, v, and the densities. Assume that the drop is small enough that air resistance is negligible.

   (b) Solve the EOM from part (a) for $v(t)$ under the assumption that $r(t) \sim r_0$ is approximately constant. *Hint:* you will find it useful to compare to assignment 1.

   (c) Based on your solution to part (b), argue that the $r(t) \sim r_0$ approximation is self-consistent for times $t \ll \sqrt{\rho_0 r_0/\rho_1 g}$. Note that your solution to $v(t)$ should be approximately linear in time for these small times.

3. **Harmonic Oscillator with Friction**

   Consider a harmonic oscillator with kinetic friction, rather than the damping we discussed in class.

   (a) Argue that Newton's law can be written as

   $$\ddot{x} + \frac{k}{m}x + \mu_k g\Theta(\dot{x}) - \mu_k g\Theta(-\dot{x}) = 0 \ , \tag{2}$$

   where Θ is the Heaviside step function, which is equal to 1 for positive argument and equal to 0 otherwise.

   (b) Solve (2) for the position of the oscillator numerically using Maple software. We will choose time units in which $m/k = 1$, so the period of the oscillator without friction is 2π. Start by choosing $\mu_k g = 0.01$, take initial conditions $x(0) = 1, \dot{x}(0) = 0$, and plot your solution, using the following code:

```
with(plots):
eqns := {(D[1, 1](x))(t)+x(t)+0.01*(Heaviside((D(x))(t))- Heaviside(-(D(x))(t)))
= 0, x(0) = 1, (D(x))(0) = 0};
soln:=dsolve(eqns,numeric,range=0..13)
odeplot(soln)
```

Then find solutions for the same initial conditions and $\mu_k g = 0.05, 0.1, 0.2$. Finally, take $\mu_k g = 0.05$ and plot the solution for $x(0) = 0.5$ and 2. Attach a printout of your Maple code and results.

(c) *inspired by FC C3.5* Now suppose oscillating mass is on a moving belt of velocity $+u$, so the friction force points opposite the relative velocity $v - u$. Redo your numerical solutions of the previous part for $\mu_k g = 0.05, x(0) = 1, \dot{x}(0) = 0$ and $u = 0.5, 1, 2$. Attach a printout of your Maple code and results.

4. **Damped Oscillators** *from Taylor*

In this problem, consider a damped oscillator as discussed in the lecture notes.

(a) Show that the position of a critically damped or overdamped oscillator can never pass through the equilibrium position $x = 0$ more than once (after having set initial conditions at $t = 0$).

(b) The underdamped oscillator does in fact oscillate, but the solution is not a pure sine wave. However, we can define the period as the time between successive maxima or as twice the time between successive zeros of $x(t)$. Show that either definition gives a period $2\pi/\bar{\omega}$, where $\bar{\omega} = \sqrt{\omega_0^2 - \gamma^2}$ as defined in the notes.

5. **The Simple Pendulum Beyond Linearity**

The simple pendulum of length l and mass m can be described by kinetic energy $T = ml^2\dot{\theta}^2/2$ and potential energy $V = mgl(1 - \cos\theta) = 2mgl\sin^2(\theta/2)$ (the last follows from angle addition formulae), where θ is the angle of the pendulum from downward. The pendulum is initially at rest at a maximum angle θ_0.

(a) Using conservation of energy, show that the time t and position θ are related by the integral

$$t = \frac{1}{2}\sqrt{\frac{l}{g}} \int_{\theta}^{\theta_0} \frac{d\theta'}{\sqrt{\sin^2(\theta_0/2) - \sin^2(\theta'/2)}} \tag{3}$$

as the pendulum falls from θ_0 to $\theta = 0$.

(b) Suppose $\theta_0 \ll 1$ as in the usual case. Expand the sine functions in (3) to lowest order and carry out the integral to show that $\theta(t) = \theta_0 \cos(\sqrt{g/l}\, t)$ as expected.

(c) Now consider the opposite limit with θ_0 and θ both close to π (directly overhead) for early times. Define $\theta = \pi - \alpha$, etc, and expand the integral to lowest order in terms of α, α_0. Carry out the integral using a hyperbolic trig substitution to show that the angular displacement starts out growing exponentially.

(d) If we take $\theta = 0$, the integral for the time gives one quarter of the full period of the pendulum. Change integration variables to ϕ, where $\sin\phi = \sin(\theta'/2)/\sin(\theta_0/2)$ to write the period in terms of the *complete elliptic integral of the first kind* $K(k)$, where $k =$

$\sin(\theta_0/2)$ for us and

$$K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \ . \tag{4}$$

(e) Having an answer in terms of a special function like the elliptic integral can be useful, since a lot is known about many special functions. For example, software like Maple has many functions available, so we could plot the period as a function of θ_0. In this case, use the first term of the asymptotic expansion (19.12.1) in the *Digital Library of Mathematical Functions* (`http://dlmf.nist.gov/`) to show that the period grows like $-\ln(\cos(\theta_0/2))$ as $\theta_0 \to \pi$. Note that $k' = \sqrt{1 - k^2}$ in that formula, the Pochhammer symbol $(1/2)_m$ is 1 for $m = 0$, and the definition of $d(0)$ is given below (19.12.3).