Intermediate Mechanics PHYS-3202
Final Exam

Dr. Andrew Frey

6 Dec 2019

Instructions:

- Do not turn over until instructed.
- You will have 3 hours to complete this exam.
- No electronic devices, hardcopy notes, or books are allowed.
- INSTRUCTIONS REGARDING TEST LENGTH WILL GO HERE.
- **Answer all questions briefly and completely.**
- **Only the lined pages of your exam book will be graded. Use the blank pages for scratch work only.**

Useful Formulae:

- Friction: $F_{static} \leq \mu_S N$, $F_{kinetic} = \mu_K N$ directed against opposing force or relative velocity
- Air/fluid resistance: $F = -\lambda v^{n-1}\vec{v}$, $\lambda \equiv m\gamma$
  - Velocity is given with respect to air/fluid
  - Terminal velocity in uniform gravity $v = (mg/\lambda)^{1/n}$
  - Solution in uniform gravity for linear air resistance

$$x = \frac{v_{x,0}}{\gamma}\left(1 - e^{-\gamma t}\right) \ , \quad z = \left(\frac{v_{z,0}}{\gamma} + \frac{g}{\gamma^2}\right)\left(1 - e^{-\gamma t}\right) - \frac{gt}{\gamma}$$

  - Solution for vertical motion in uniform gravity for quadratic air resistance

$$v = -\sqrt{\frac{g}{\gamma}}\tanh\left(\sqrt{\gamma g}\, t\right) \ , \quad z = z_0 - \frac{1}{\gamma}\ln\left[\cosh\left(\sqrt{\gamma g}\, t\right)\right]$$

  with $z = z_0, v = 0$ at $t = 0$, z increasing upward

- Thrust $-\dot{m}u$ for u exhaust speed, $v = v_0 + u\ln(m_0/m)$
- Harmonic Oscillators
  - Linear restoring force $F = -kx \equiv -m\omega_0^2 x$, linear damping force $F = -\lambda\dot{x} = -2m\gamma\dot{x}$
  - Potential $V = kx^2/2$
  - Independent solutions $x = Ae^{pt}$ where $p = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$
  - For driving force $F(t) = Fe^{i\omega t}$, solution is transients plus

$$x(t) = Ae^{i\omega t - i\theta} \ , \quad A = \frac{F/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} \ , \quad \tan\theta = \frac{2\gamma\omega}{\omega_0^2 - \omega^2}$$

- Isotropic oscillator in 3D has restoring force $\vec{F} = -k\vec{r}$, potential $V = kr^2/2$
- Can be anisotropic in 3D

- General 3D concepts
  - Centripetal acceleration $a_r = -v^2/r$ for circular motion
  - Angular momentum $\vec{J} = \vec{r} \times \vec{p}$, torque $\vec{\tau} = \vec{r} \times \vec{F}$
  - Effective potential for central force $U = V + J^2/2mr^2$
  - For a central force, motion is in a plane $\perp \vec{J}$
  - For a central force, Kepler's second law $dA/dt = J/2m$ constant

- Inverse square force $\vec{F} = k\hat{r}/r^2$, $V = k/r$
  - Coulomb $k = q_1 q_2/4\pi\epsilon_0$, gravity $k = -Gm_1 m_2$
  - Orbit $\ell = r\,(e\cos\phi \pm 1)$, $\ell = J^2/m|k|$, $e = \sqrt{1 + 2J^2E/mk^2}$
  - Elliptical orbit: semimajor axis $a = \ell/(1 - e^2)$, semiminor axis $b = \ell/\sqrt{1 - e^2}$
    Kepler's 3rd law $T^2 = 4\pi^2 a^3/GM$ for gravity
  - Hyperbolic orbit $a = \ell/(e^2 - 1)$ with impact parameter $b = \ell/\sqrt{e^2 - 1}$

- Scattering
  - Mean free path $\lambda = 1/n\sigma$, n =number density, σ =cross section
  - Hard sphere scattering $b = R\cos(\theta/2)$, $d\sigma/d\Omega = R^2/4$
  - Rutherford scattering $b = (k/mv^2)\cot(\theta/2)$, $d\sigma/d\Omega = (1/4)(k/mv^2)^2(1/\sin^4(\theta/2))$

- Noninertial Frames with Accelerating Origin at $\vec{R}$
  - General motion of origin: fictitious force $\vec{F} = -md^2\vec{R}/dt^2$
  - Time derivative in inertial axes vs rotating axes $d\vec{a}/dt = \dot{\vec{a}} + \vec{\omega} \times \vec{a}$, angular velocity $\vec{\omega}$
  - For $\vec{R}$ rotating around same inertial origin:
    * Transverse force $-m\dot{\vec{\omega}} \times (\vec{R} + \vec{r})$
    * Centrifugal force $-m\vec{\omega} \times (\vec{\omega} \times (\vec{R} + \vec{r})) = m\omega^2(\vec{R} + \vec{r}) - m\vec{\omega}(\vec{\omega} \cdot (\vec{R} + \vec{r}))$
    * Coriolis force $-2m\vec{\omega} \times \dot{\vec{r}}$

- Inertia Tensor
  - Mass, center of mass position, inertia tensor ($dm = d^3\vec{r}\,\rho$)

$$M = \int dm\ ,\quad M\vec{R} = \int dm\,\vec{r}\ ,\quad I_{ij} = \int dm\left(r^2\delta_{ij} - r_i r_j\right)$$

  - Moments of inertia are diagonal components, products off-diagonal
    I_{zz} is around z axis, etc
  - Principal axes and moments are eigenvectors and eigenvalues
  - Parallel Axis Theorem $I'_{ij} = I^{CM}_{ij} + M(R^2\delta_{ij} - R_i R_j)$, I^{CM}_{ij} is around center of mass
  - Angular momentum $J_i = \sum_j I_{ij}\omega_j$, kinetic energy $T = \sum_{ij} I_{ij}\omega_i\omega_j/2$

- Rigid Bodies
  - Inertial axes $\hat{\imath}, \hat{\jmath}, \hat{k}$, body-fixed principal axes $\hat{e}_1, \hat{e}_2, \hat{e}_3$
  - Rolling without slipping $v = \omega R$
  - Euler equations in rotating body frame $\dot{\vec{J}} + \vec{\omega} \times \vec{J} = \vec{\tau}$; in principal axes:

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2 = \tau_1\ ,\quad I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 = \tau_2\ ,\quad I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 = \tau_3$$

- Euler angles
  - Rotate by ϕ around $\hat{k}$
  - Rotate by θ around line of nodes ($\hat{x}$) to define $\hat{x}, \hat{y}, \hat{z}$ axes
  - Rotate by ψ around principal axis $\hat{e}_3$
  - $\hat{x}, \hat{y}, \hat{z}$ axes rotate with angular velocity $\vec{\eta} = \dot{\theta}\hat{x} + \dot{\phi}\sin\theta\hat{y} + \dot{\phi}\cos\theta\hat{z}$
  - Angular velocity

$$\vec{\omega} = \vec{\eta} + \dot{\psi}\hat{z} = (\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\hat{e}_1 + (\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)\hat{e}_2 + (\dot{\phi}\cos\theta + \dot{\psi})\hat{e}_3$$

- Symmetric object free rotation "wobble" rate $\dot{\phi} = I_3\omega_3/I\cos\theta$
- Effective potential for nutation of symmetric object in gravity

$$V(\theta) = \frac{1}{2}\frac{(J_k - J_z\cos\theta)^2}{I\sin^2\theta} + MgR\cos\theta$$

- Astronomical data
  - Earth latitude $\lambda = \pi/2 - \theta$, $\theta =$ polar angle, colatitude
  - $\vec{g}$ defined by convention, you can use $g = GM_\oplus/R_\oplus^2 \approx 9.8$ m/s^2
  - Earth mass $M_\oplus = 6.0 \times 10^{24}$ kg, equatorial radius $R_\oplus = 6400$ km
  - Earth orbit semimajor axis $a_\oplus = 1$ au$= 1.5 \times 10^8$ km, period $T_\oplus = 1$ yr$= 3.2 \times 10^7$ s
  - Solar mass $M_\odot = 2.0 \times 10^{30}$ kg, equatorial radius $R_\odot = 7.0 \times 10^8$ m

- Vector Calculus
  - Triple products $(\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{b} \times \vec{c}) \cdot \vec{a}$, $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$
  - Gradient operator

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$

    Divergence $\vec{\nabla} \cdot \vec{A}$, Curl $\vec{\nabla} \times \vec{A}$, $\vec{\nabla} \times \vec{\nabla} f = 0$, $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$
  - Cylindrical coordinates $x = \rho\cos\varphi, y = \rho\sin\varphi, z = z$

$$ds^2 = d\rho^2 + \rho^2 d\varphi^2 + dz^2 \ , \ \ d^3\vec{r} = \rho d\rho d\varphi dz \ , \ \ \vec{v} = \dot{\rho}\hat{\rho} + \rho\dot{\varphi}\hat{\varphi} + \dot{z}\hat{z} \ , \ \ \vec{\nabla} f = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \varphi}\hat{\varphi} + \frac{\partial f}{\partial z}\hat{z}$$

  - Spherical polar coordinates $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \ , \ \ d^3\vec{r} = r^2 dr\sin\theta d\theta d\phi \ , \ \ \vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi}$$

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$$

- Other Math
  - Ellipse and hyperbola in Cartesian coordinates (centered at origin)

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$$

    semi-major axis a, semi-minor axis b
  - Trigonometric identities

$$\cos^2\theta + \sin^2\theta = 1 \ , \ \ \sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin(\alpha+\beta) \ , \ \ \cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos(\alpha+\beta)$$

$$d\sin\theta/d\theta = \cos\theta \ , \ \ d\cos\theta/d\theta = -\sin\theta \ , \ \ e^{i\theta} = \cos\theta + i\sin\theta$$

  - Matrix A, eigenvalues λ, eigenvectors $\vec{x}$: $A\vec{x} = \lambda\vec{x}$
    Characteristic equation $\det(A - \lambda 1) = 0$