Intermediate Mechanics PHYS-3202
In-Class Test

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Instructions:

- Do not turn over until instructed.
- You will have 50 minutes to complete this test.
- No electronic devices or hardcopy notes are allowed.
- INSTRUCTIONS REGARDING TEST LENGTH WILL GO HERE.
- **Answer all questions briefly and completely.**
- **Only the lined pages of your exam book will be graded. Use the blank pages for scratch work only.**

Useful Formulae:

- Friction: $F_{static} \leq \mu_S N$, $F_{kinetic} = \mu_K N$ directed against opposing force or relative velocity
- Air/fluid resistance: $F = -\lambda v^{n-1}\vec{v}$, $\lambda \equiv m\gamma$
  - Velocity is given with respect to air/fluid
  - Terminal velocity in uniform gravity $v = (mg/\lambda)^{1/n}$
  - Solution in uniform gravity for linear air resistance

$$x = \frac{v_{x,0}}{\gamma}\left(1 - e^{-\gamma t}\right) \ , \quad z = \left(\frac{v_{z,0}}{\gamma} + \frac{g}{\gamma^2}\right)\left(1 - e^{-\gamma t}\right) - \frac{gt}{\gamma}$$

  - Solution for vertical motion in uniform gravity for quadratic air resistance

$$v = -\sqrt{\frac{g}{\gamma}}\tanh\left(\sqrt{\gamma g}\, t\right) \ , \quad z = z_0 - \frac{1}{\gamma}\ln\left[\cosh\left(\sqrt{\gamma g}\, t\right)\right]$$

    with $z = z_0, v = 0$ at $t = 0$, z increasing upward

- Thrust $-\dot{m}u$ for u exhaust speed, $v = v_0 + u\ln(m_0/m)$
- Harmonic Oscillators
  - Linear restoring force $F = -kx \equiv -m\omega_0^2 x$, linear damping force $F = -\lambda\dot{x} = -2m\gamma\dot{x}$
  - Potential $V = kx^2/2$
  - Independent solutions $x = Ae^{pt}$ where $p = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$
  - For driving force $F(t) = Fe^{i\omega t}$, solution is transients plus

$$x(t) = Ae^{i\omega t - i\theta} \ , \quad A = \frac{F/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} \ , \quad \tan\theta = \frac{2\gamma\omega}{\omega_0^2 - \omega^2}$$

- Isotropic oscillator in 3D has restoring force $\vec{F} = -k\vec{r}$, potential $V = kr^2/2$
- Can be anisotropic in 3D

- General 3D concepts
  - Centripetal acceleration $a_r = -v^2/r$ for circular motion
  - Angular momentum $\vec{J} = \vec{r} \times \vec{p}$, torque $\vec{\tau} = \vec{r} \times \vec{F}$
  - Effective potential for central force $U = V + J^2/2mr^2$
  - For a central force, motion is in a plane $\perp \vec{J}$
  - For a central force, Kepler's second law $dA/dt = J/2m$ constant

- Inverse square force $\vec{F} = k\hat{r}/r^2$, $V = k/r$
  - Coulomb $k = q_1 q_2/4\pi\epsilon_0$, gravity $k = -Gm_1m_2$
  - Orbit $\ell = r\,(e\cos\phi \pm 1)$, $\ell = J^2/m|k|$, $e = \sqrt{1 + 2J^2E/mk^2}$
  - Elliptical orbit: semimajor axis $a = \ell/(1-e^2)$, semiminor axis $b = \ell/\sqrt{1-e^2}$
    Kepler's 3rd law $T^2 = 4\pi^2 a^3/GM$ for gravity
  - Hyperbolic orbit $a = \ell/(e^2-1)$ with impact parameter $b = \ell/\sqrt{e^2-1}$

- Scattering
  - Mean free path $\lambda = 1/n\sigma$, n =number density, σ =cross section
  - Hard sphere scattering $b = R\cos(\theta/2)$, $d\sigma/d\Omega = R^2/4$
  - Rutherford scattering $b = (k/mv^2)\cot(\theta/2)$, $d\sigma/d\Omega = (1/4)(k/mv^2)^2(1/\sin^4(\theta/2))$

- Vector Calculus
  - Triple products $(\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{b} \times \vec{c}) \cdot \vec{a}$, $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$
  - Gradient operator
  $$\vec{\nabla} = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$
    Divergence $\vec{\nabla} \cdot \vec{A}$, Curl $\vec{\nabla} \times \vec{A}$, $\vec{\nabla} \times \vec{\nabla} f = 0$, $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$
  - Cylindrical coordinates $x = \rho\cos\varphi, y = \rho\sin\varphi, z = z$
  $$ds^2 = d\rho^2 + \rho^2 d\varphi^2 + dz^2\ ,\ \ d^3\vec{r} = \rho d\rho d\varphi dz\ ,\ \ \vec{v} = \dot{\rho}\hat{\rho} + \rho\dot{\varphi}\hat{\varphi} + \dot{z}\hat{z}\ ,\ \ \vec{\nabla} f = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \varphi}\hat{\varphi} + \frac{\partial f}{\partial z}\hat{z}$$
  - Spherical polar coordinates $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$
  $$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\ ,\ \ d^3\vec{r} = r^2 dr\sin\theta d\theta d\phi\ ,\ \ \vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi}$$
  $$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$$

- Ellipse and hyperbola in Cartesian coordinates (centered at origin)
$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$$
semi-major axis a, semi-minor axis b