Cases of Rigid Body Motion

- Some general comments

  • Suppose we have a general origin with CM at $\vec{R}$
    + Angular momentum
    $$\vec{J} = M\vec{R}\times d\vec{R}/dt + \overleftrightarrow{I}_{cm}\vec{\omega}_{cm}$$
    where $\overleftrightarrow{I}_{cm}$ and $\vec{\omega}_{cm}$ are the inertia tensor and angular velocity around center of mass.
    Derivation is similar to parallel axis theorem
    + Kinetic energy
    $$T = \frac{1}{2}M(d\vec{R}/dt)^2 + \frac{1}{2}\vec{\omega}_{cm}\cdot(\overleftrightarrow{I}_{cm}\vec{\omega}_{cm})$$
    + In both cases, we used that the position of a point on the body is
    $$\vec{r}\,' = \vec{R} + \vec{r}$$
    where $\vec{r}$ = position relative to the center of mass
    + Motion splits into motion of a point-like object at center of mass plus body rotation
    + In particular, the total force is
    $$M\,d^2\vec{R}/dt^2 = \int dm\,\frac{d^2\vec{r}}{dt^2} = \int d\vec{F}_{int} + \int d\vec{F}_{ext} = \vec{F}_{ext}$$
    so total external force gives acceleration of center of mass.

  • Now consider an accelerating origin at position $\vec{R}$ w.r.t. an inertial origin.
    + Inertial position $\vec{r}\,' = \vec{R} + \vec{r}$ ← accelerating frame position
    $\vec{r}_{cm}$ = center of mass position in accelerating frame.
    + The angular momenta in the 2 frames are related by
    $$\vec{J}\,' = M\vec{R}\times\frac{d\vec{r}_{cm}}{dt} + M\vec{r}_{cm}\times\frac{d\vec{R}}{dt} + \vec{J}$$

+ In the inertial frame, we have 2 expressions for torque via $\vec{\tau}' = d\vec{J}'/dt$ (all forces external)

Then

$$\vec{\tau}' = \int \vec{r}' \times d\vec{F}' = \vec{R} \times \vec{F}' + \int \vec{r} \times (d\vec{F} + \underbrace{dm \frac{d^2\vec{R}}{dt^2}}_{\text{fictitious}})$$

$$= \vec{R} \times \vec{F}' + M \vec{r}_{cm} \times \frac{d^2\vec{R}}{dt^2} + \vec{\tau}$$

Note that $\vec{\tau}$ includes fictitious forces. The 1st term is just due to change of location of origin.

+ Meanwhile,

$$\frac{d\vec{J}'}{dt} = M\vec{R} \times \frac{d^2\vec{r}_{cm}'}{dt^2} + M\vec{r}_{cm} \times \frac{d^2\vec{R}}{dt^2} + \frac{d\vec{J}}{dt}$$

b/c some cross terms cancel.

+ If we compare, $d\vec{J}/dt = \vec{\tau}$ still but only including fictitious forces in calculating the torque. Since the fictitious force is uniform and $\propto$ mass, it acts at the center of mass for the torque calculation

• Describing the motion from the rotating body frame: Euler's equations

+ Once again, with respect to inertial axes,

$$\frac{d\vec{J}}{dt} = \vec{\tau} \leftarrow \text{external torque}$$

+ But if we use any body frame axes, like the principal axes, that's a rotating frame w/ angular velocity $\vec{\omega}$. In our previous notation

$$\frac{d\vec{J}}{dt} = \dot{\vec{J}} + \vec{\omega} \times \vec{J} = \vec{\tau}$$

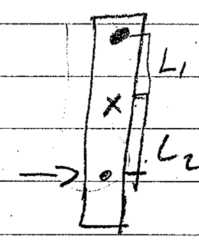
These are Euler's equations

+ For a rigid body, the inertia tensor w.r.t. the body does not change. In terms of the principal axes $\hat{e}_1, \hat{e}_2, \hat{e}_3$, we have

$I_1 \dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = \tau_1$
and permutations

+ Can be difficult to use if torque is constant w.r.t fixed inertial axes.

— More examples of planar motion



• "Baseball Bat Theorem"
+ An object can pivot around a point L_1 from the center of mass. A sharp force acts on the object at a different point L_2 from the center of mass. When is the force on the pivot zero?

+ As a result of the impulse, the momentum of the object immediately after the strike is $P = MV_{cm} = ML_1\omega$, where ω = new angular velocity.

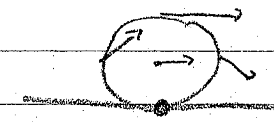
+ Meanwhile the impulse acts with a torque creating angular momentum $P(L_1+L_2) = I\omega$ around the pivot, where I = moment of inertia.

+ Dividing these equations, we see
$$I = ML_1^2 + ML_1L_2$$
for no force on the pivot. Using the parallel axis theorem, $I = I_{cm} + ML_1^2$ where I_{cm} is the moment around the CM. This yields
$$ML_1L_2 = I_{cm}$$
as the condition on L_1 + L_2

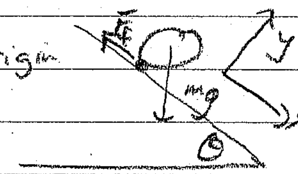
+ L_2 is the "center of percussion" for the pivot at L_1. Baseball players call it the "sweet spot" to avoid having the bat exert force on their hands.

- Rolling with and without slipping for objects with a circular profile

+ Rolling is translational + rotational motion. + Without slipping, the contact point is always instantaneously at rest w.r.t. the surface. This means the linear velocity $v = \omega R$, where ω is the angular velocity of the rotation + R the radius of the circular profile. There is static friction.



+ Ex: Rolling on a slope, center of mass as origin. Linear motion is determined by



$$M\ddot{x} = Mg\sin\theta - \mu_s Mg\cos\theta \quad \text{as usual.}$$

Meanwhile, in center of mass frame, torque from friction.

So $\quad I_{cm}\dot{\omega} = \mu_s Mg\cos\theta\, R$

(assuming the axis of rotation is principal)

But then $\omega = \dot{x}/R$, so

$$\ddot{x} = \frac{g\sin\theta}{1 + I_{cm}/MR^2}$$

+ Same thing, using the contact point as origin. This time, the torque is

$$I\dot{\omega} = Rmg\sin\theta \quad w/\ I = I_{cm} + MR^2$$

Again, where x = center of mass position, $\dot{\omega} = \ddot{x}/R$

We get the same answer.

+ Suppose the center of mass moves a distance x on the incline. What's the speed? Energy conservation:

$$mgx\sin\theta = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}I_{cm}(v/R)^2$$

+ With slipping, there is kinetic friction: $v \neq \omega R$
Consider the same example.

$$\ddot{x} = g\sin\theta - \mu_k g\cos\theta$$
$$\dot{\omega} = \mu_k M g R\cos\theta / I_{cm}$$

So ω and $v = \dot{x}$ have a constant ratio

— Force-free Motion = No torques

• Stability of rotation around a principal axis

+ Suppose the initial motion of a freely-rotating object is given by $\vec{\omega} = \omega_1\hat{e}_1 + \omega_2\hat{e}_2 + \omega_3\hat{e}_3$ with $\omega_1, \omega_2 \ll \omega_3$ where $\hat{e}_i$ are principal axes. Rotation mostly around $\hat{e}_3$

+ Since $\omega_1 + \omega_2$ are small, neglect $\omega_1\omega_2$ in Euler equations. They become

$$I_3\dot{\omega}_3 \approx 0, \quad I_1\dot{\omega}_1 = (I_2 - I_3)\omega_3\omega_2, \quad I_2\dot{\omega}_2 = (I_3 - I_1)\omega_1\omega_3$$

+ So ω_3 is basically constant. By differentiating,

$$\ddot{\omega}_1 \approx \frac{(I_3 - I_1)(I_2 - I_3)}{I_1 I_2}\omega_3^2\omega_1$$

We guess a (complex) solution $\omega_1 = Ae^{i\Omega t}$
We get

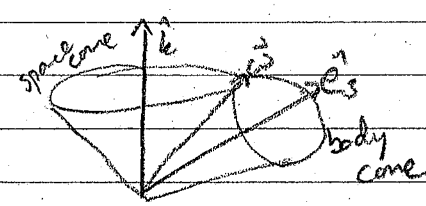
$$\Omega^2 = \frac{(I_3 - I_1)(I_3 - I_2)}{I_1 I_2}\omega_3^2$$

and ω_2 takes the same functional form.

+ If $I_3 > I_1, I_3 > I_2$ or $I_3 < I_1, I_3 < I_2$ (ie, largest or smallest moment), Ω is real.
That means $\omega_1 + \omega_2$ oscillate + stay small.
If $I_1 < I_3 < I_2$ or $I_2 < I_3 < I_1$ (I_3 is middle moment), Ω is imaginary, so $\omega_1 + \omega_2$ grow.

+ In other words, rotation around the axis with largest or smallest is stable. The middle one is unstable.
If 2 moments have equal moments, rotation around them is unstable, but the other is stable (check).

- Free Rotation of a Symmetric Top (Rotating Object)
  - Consider a symmetric rigid body (top) with principal moments $I \equiv I_1 = I_2$ and I_3. It has angular velocity $\vec{\omega} = \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3$ in terms of principal axes
  - The Euler equations are
    $$I_3 \dot{\omega}_3 = 0, \quad I\dot{\omega}_1 = (I - I_3)\omega_3\omega_2, \quad I\dot{\omega}_2 = (I_3 - I)\omega_3\omega_1$$
    we immediately see $\omega_3 = \text{const}$
  - We can define $\Omega = (I_3 - I)\omega_3 / I$, so
    $$\dot{\omega}_1 + \Omega\omega_2 = 0, \quad \dot{\omega}_2 - \Omega\omega_1 = 0$$
  - We've seen this type of equation before (Lorentz force)
    Answer is
    $$\omega_1 = A\cos(\Omega t + \delta), \quad \omega_2 = A\sin(\Omega t + \delta)$$
  - Describe this motion: ω_1 & ω_2 describe circular motion in the plane of $\hat{e}_1, \hat{e}_2$. Further, the magnitude is constant. This means $\vec{\omega}$ precesses around $\hat{e}_3$ with frequency Ω. $\vec{\omega}$ traces a cone called the <u>body cone</u> around $\hat{e}_3$
  - Alternately, note that $\vec{J}$ is conserved. Choose inertial axes so $\vec{J} = J\hat{k}$. Further, the conserved kinetic energy $T = \frac{1}{2}\vec{\omega}\cdot(\overleftrightarrow{I}\vec{\omega}) = \frac{1}{2}\vec{\omega}\cdot\vec{J}$, so the angle between is constant. $\Rightarrow \vec{\omega}$ precesses around $\hat{k}$ also. Then $\vec{\omega}$ traces out a cone called the <u>space cone</u> around $\hat{k}$

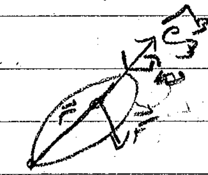


  - The earth's rotation is actually not lined up with its principal axis, so it precesses like this. For the earth, $I \approx I_3$, so Ω is small, leading to a ~300 day period. It's actually longer b/c earth is not quite rigid

— Precession from a Small Torque

- Consider a body rotating around an axis with 1 fixed point but otherwise free to rotate

  + To avoid free-body precession, assume the axis is principal axis $\hat{e}_3$. Further assume a small force $\vec{F}$ acts at point $\vec{r}$ on the axis.



  + Assuming the force is small, the motion of the axis will be slow compared to the rotation around the axis. $\Rightarrow$ We can treat the motion as a slow change in direction of $\hat{e}_3$ (and $\vec{\omega}$) and not rotation around $\hat{e}_1$ or $\hat{e}_2$

  + The equation of motion is then
$$\frac{d\vec{J}}{dt} \approx I_3 \frac{d\vec{\omega}}{dt} = \vec{r} \times \vec{F}$$
Because $\vec{r} \parallel \vec{\omega}$, $\vec{\omega}$ changes direction only, $\perp$ to $\vec{F}$!

  + Suppose the force is gravity $-Mg\hat{k}$. The spinning object is a top or gyroscope. If the center of mass is a distance R from the fixed point, $\vec{\omega} = \omega \hat{e}_3$, $\vec{r} = R\hat{e}_3$, so
$$I_3 \omega \frac{d\hat{e}_3}{dt} = -MgR\, \hat{e}_3 \wedge \hat{k} \Rightarrow \frac{d\hat{e}_3}{dt} = \vec{\Omega} \wedge \hat{e}_3$$
where $\vec{\Omega} = (MgR/I_3\omega)\,\hat{k}$.
This is again precession around $\hat{k}$ at frequency Ω.

  + Note that $\Omega \propto (I_3 \omega)^{-1}$! Rapidly spinning, wide objects barely precess. This is why gyroscopes point in a fixed direction

  + The sun + moon exert a torque on the earth due to its slightly oblate shape. This causes precession of the equinoxes with period $\simeq$ 26000 years.