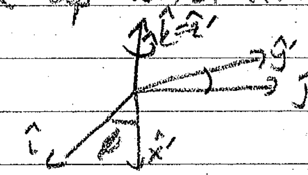
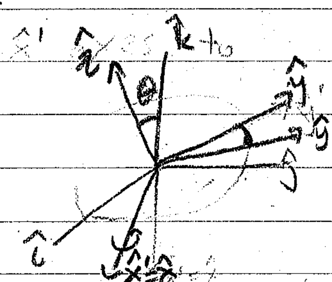
An Absolute Description of Rotation: Euler Angles

- Orientation of a rigid body
  - It takes 3 angles to describe the orientation of any object with respect to fixed axes.
    - + By far the most popular set is due to Euler.
    - + An alternative is to use quaternions (generalized complex numbers)

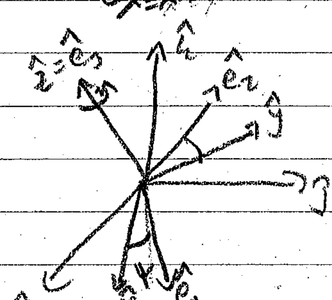
  - Euler (or Eulerian) Angles ϕ, θ, ψ.
    - + Start with a fixed set of axes $\hat{i}, \hat{j}, \hat{k}$ (unit vectors) and line up the principal axes $\hat{e}_1, \hat{e}_2, \hat{e}_3$ with them.
    - + Next, rotate the object by ϕ around $\hat{k}$. The principal axes now line up with unit vectors $\hat{x}', \hat{y}', \hat{z}'$ (with $\hat{z} = \hat{k}$)



    - + Now rotate by angle θ around the $\hat{x}'$ axis to line the principal axes up with new $\hat{x}, \hat{y}, \hat{z}$ axes ($\hat{x} = \hat{x}'$)



    - + Finally, rotate by angle ψ around the $\hat{z}$ axis to get the final principal body axis positions. Note $\hat{z} = \hat{e}_3$.



    - + There are 3 important axes in this process: $\hat{k}$, $\hat{x} = \hat{x}'$, and $\hat{e}_3$. The $\hat{x}$ axis is called the line of nodes
    - + Note: I set the line of nodes as $\hat{x}$ (like FC, TM books). KB + Taylor books use $\hat{y}$ as the line of nodes.

- Motion in terms of Euler angles
  - To describe rotation (beyond just orientation), ϕ, θ, ψ change in time.
  - The object, aka the rotating frame of principal axes, has angular velocity vector
$$\vec{\omega} = \dot{\phi}\hat{k} + \dot{\theta}\hat{x} + \dot{\psi}\hat{e}_3$$
  - We will also find it useful to consider the rotating frame of the $\hat{x}, \hat{y}, \hat{z}$ axes, which has angular velocity vector
$$\vec{\eta} = \dot{\phi}\hat{k} + \dot{\theta}\hat{x}$$

  - It's useful to know how to write the unit vectors along each set of axes
$$\hat{k} = \cos\theta\,\hat{z} + \sin\theta\,\hat{y} = \cos\theta\,\hat{e}_3 + \sin\theta(\cos\psi\,\hat{e}_2 + \sin\psi\,\hat{e}_1)$$
$$\hat{x} = \cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath} = \cos\psi\,\hat{e}_1 - \sin\psi\,\hat{e}_2$$
$$\hat{e}_3 = \hat{z} = \cos\theta\,\hat{k} + \sin\theta(-\cos\phi\,\hat{\jmath} + \sin\phi\,\hat{\imath})$$
(Further details in TM)

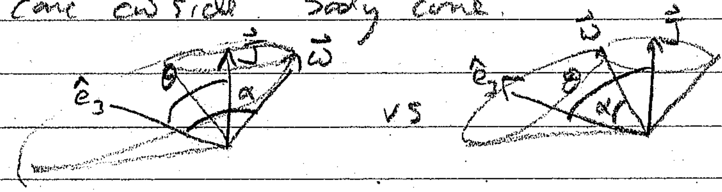
  - In particular, we can write
$$\vec{\eta} = \dot{\theta}\hat{x} + \dot{\phi}\sin\theta\,\hat{y} + \dot{\phi}\cos\theta\,\hat{z}$$
and
$$\vec{\omega} = \dot{\theta}\hat{x} + \dot{\phi}\sin\theta\,\hat{y} + (\dot{\phi}\cos\theta + \dot{\psi})\hat{z}$$
$$= (\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\hat{e}_1 + (\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)\hat{e}_2 + (\dot{\phi}\cos\theta + \dot{\psi})\hat{e}_3$$

  - Real shorthand; we sometimes define
$$\omega_3 = \vec{\omega}\cdot\hat{e}_3 = \dot{\phi}\cos\theta + \dot{\psi}$$
as the "spin" b/c we often consider an object rotating rapidly around $\hat{e}_3$ with $\hat{e}_3$ also moving in orientation.

  - For a symmetric object with $I_1 = I_2$, any vector in the $\hat{e}_1, \hat{e}_2$ plane describes a principal axis (they are all eigenvectors of $\overleftrightarrow{I}$). In this case, we can use $\hat{x}, \hat{y}, \hat{z} = \hat{e}_3$ as principal axes (even though the object rotates past them)

- Revisiting precession with Euler Angles: Symmetric Objects

- Free body precession
  - + Recall that a freely rotating symmetric object has angular velocity $\vec{\omega}$ that precesses around $\hat{e}_3$ with frequency $\Omega = (I_3 - I)\omega_3 / I$ and around $\vec{J} = J\hat{k}$
  - + We'd like to know the relation between the body + space cones and the "wobble" rate of $\vec{\omega}$ around $\vec{J}$.
  - + We note $\vec{J} = J\sin\theta\,\hat{y} + J\cos\theta\,\hat{z}$, which means $\vec{J}\cdot\hat{x} = J_x = I\dot{\theta} = 0$, so θ is fixed b/c $\vec{J}$ is conserved. That means $\omega_x = 0$, so write $\vec{\omega} = \omega\sin\alpha\,\hat{y} + \omega\cos\alpha\,\hat{z}$. The difference between θ + α is due to $\dot{\psi}$. Also, $\dot{\phi}, \dot{\psi}$ const.
  - + The ratio
$$\frac{J_y}{J_z} = \tan\theta = \frac{I}{I_3}\tan\alpha \quad \text{b/c } \vec{J} = \overleftrightarrow{I}\vec{\omega}.$$
We see $\alpha > \theta$ for $I_3 > I$ (wide object), meaning the body cone contains the space cone.
Meanwhile $I_3 < I$ (skinny object) $\Rightarrow \alpha < \theta \Rightarrow$ space cone outside body cone.

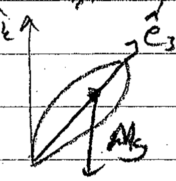


  - + The wobble rate (angular velocity of precession of $\hat{e}_3$ around $\hat{k}$) is $\dot{\phi}$ by the definition of the Euler angle.
One way to get this is to compare ω_y components
$$\omega_y = \dot{\phi}\sin\theta = \omega\sin\alpha.$$
Can find
$$\dot{\phi} = \omega\left[1 + \left(\frac{I_3^2}{I^2} - 1\right)\cos^2\alpha\right]^{1/2} = \frac{I_3}{I}\omega_3/\cos\theta$$
  - + We recall that the precession rate (of $\vec{\omega}$ around $\hat{e}_3$) is $\Omega = (I_3 - I)\omega_3 / I = (I_3/I - 1)\,\omega\cos\alpha$.
But by looking at ω_1 and ω_2 components with $\dot{\theta} = 0$, we get $\Omega = \dot{\psi}$, same as rotation of $\hat{e}_1, \hat{e}_2$ around $\hat{e}_3$.

+ We mentioned before that earth has free precession with $\Omega \approx 300$ days. The wobble rate $\dot\phi \sim 0.997$ days since $I_3 \sim I_1$.

• Precession Due to a Force: Revisiting the fixed point top

+ Consider a symmetric top fixed at one end with gravity acting at the center of mass. Precession rate $= \dot\phi$



+ If the distance from the support point to center of mass is R, the torque due to gravity is

$$\vec\tau = -MgR\,(\hat{e}_3 \times \hat{k}) = MgR\sin\theta\,(\hat{z}\times\hat{y}) = MgR\sin\theta\,\hat{x}$$

It is always directed along the line of nodes, no matter the top's orientation.

+ We therefore write the eqns. of motion in the $\hat{x}, \hat{y}, \hat{z}$ frame, which rotates w/ angular velocity $\vec\eta$.

$$\vec\tau = \frac{d\vec{J}}{dt} = \dot{\vec{J}} + \vec\eta \times \vec{J} \qquad (*)$$

+ Since the x, y components of $\vec\eta$ and $\vec\omega$ are the same and proportional to the x, y components of $\vec{J}$, the z component of $(*)$ is $\dot{J}_z = 0 \Rightarrow I_3\omega_z = I_3\omega_3 = \text{const}$, so the spin is constant. This works b/c $\hat{x}, \hat{y}$ are principal axes for the symmetric object

+ The x component is

$$MgR\sin\theta = I\ddot\theta - I\dot\phi^2\sin\theta\cos\theta + I_3\dot\phi\sin\theta\,\omega_3$$

Save this for later.

+ We can get the y component also, but that plus the z component combine to give the component in the $\hat{k}$ direction. We know that angular momentum is conserved, so write

$$(\hat{k}\cdot\vec{J}) = I\dot\phi\sin^2\theta + I_3\omega_3\cos\theta \equiv J_k = \text{constant}$$

+ The case we discussed before assumed $\theta =$ const. (This is called steady precession). In this case, the x component of the eqn of motion is

$$MgR = -I\dot{\phi}^2\cos\theta + I_3\dot{\phi}\omega_3 \Rightarrow \dot{\phi} = \frac{I_3\omega_3 \pm (I_3^2\omega_3^2 - 4MgRI\cos\theta)^{1/2}}{2I\cos\theta}$$

This configuration is possible only for $I_3^2\omega_3^2 \geq 4MgRI\cos\theta$.

+ When the force is small, $4MgRI \ll I_3^2\omega_3^2$, we expand the square root.

$$\dot{\phi} \approx I_3\omega_3/I\cos\theta \quad \text{or} \quad \dot{\phi} \approx \frac{MgR}{I_3\omega_3}$$

Our previous approximation found only the 2^nd^, or slow precession solution. The fast precession solution is rarely seen.

• Nutation: More general precessional motion

+ In general, as our symmetric top precesses, θ is not constant and in fact bobs up and down. This bobbing motion is called nutation. Consider the same system as above.

+ It is easiest to see how nutation happens from an effective potential point of view. Start with the energy written in the $\hat{x}, \hat{y}, \hat{z}$ principal axes (but not a rotating frame):

$$E = \tfrac{1}{2}I\dot{\theta}^2 + \tfrac{1}{2}I\sin^2\theta\,\dot{\phi}^2 + \tfrac{1}{2}I_3\omega_3^2 + MgR\cos\theta$$

+ We know $\omega_3 =$ const, so that term doesn't really matter. We can also see from the definition of the conserved $J_{z'}$ that

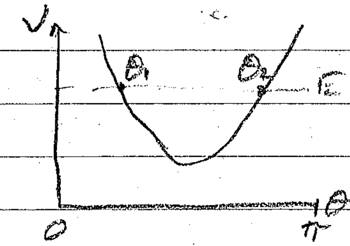
$$\dot{\phi} = \frac{J_{z'} - J_z\cos\theta}{I\sin^2\theta} \quad \text{where } J_z = \text{conserved also}$$

+ So the energy is

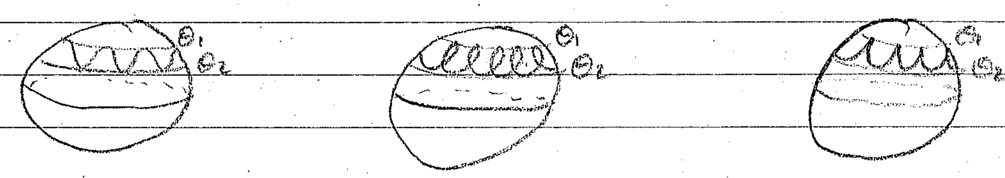
$$E = \tfrac{1}{2}I\dot{\theta}^2 + \tfrac{1}{2}I_3\omega_3^2 + \left[\frac{1}{2}\frac{(J_{z'} - J_z\cos\theta)^2}{I\sin^2\theta} + MgR\cos\theta\right]$$

$= (\text{meaningless constant}) + \frac{1}{2} I \dot{\theta}^2 + V(\theta)$
for an effective potential $V(\theta)$

+ So, depending on the energy, θ oscillates between allowed values θ_1 and θ_2



+ There are three general patterns for nutation (beyond the steady precession $\dot{\theta}=0$ case)



$\dot{\phi}$ always positive $\quad$ $\dot{\phi}$ negative near θ_1 $\quad$ $\dot{\phi} = 0$ exactly at θ_1