Larmor Precession

Charged particle in magnetic field again

- We will look at a charged particle:
  1) orbiting an oppositely charged particle (ie, in an attractive Coulomb potential)
  2) in a weak magnetic field

- This is another Lorentz force example which will tie in to our next discussion (rigid body motion)

- In some ways, this is more appropriate in quantum, but we'll look at the classical system

— Can we simplify the motion in a rotating frame?

- In an inertial frame, Newton's 2^{nd} Law is

$$m\, d^2\vec{r}/dt^2 = -k/r^2\, \hat{r} + q(d\vec{r}/dt) \times \vec{B}$$

where $k = \dfrac{qQ}{4\pi\epsilon_0}$ and $\vec{B} = B\hat{k}$

- But suppose we have a frame rotating w/ angular velocity $\vec{\omega}$

  + Acceleration $\dfrac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + 2\vec{\omega}\times\dot{\vec{r}}$

  + Lorentz force $\dfrac{d\vec{r}}{dt}\times\vec{B} = \dot{\vec{r}}\times\vec{B} - \vec{B}\times(\vec{\omega}\times\vec{r})$

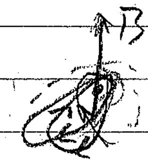
  + If we choose $\vec{\omega} = -q\vec{B}/2m$, Coriolis + "Lorentz" forces cancel, leaving

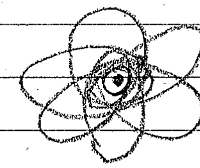
$$m\ddot{\vec{r}} = -\frac{k}{r^2}\hat{r} + \frac{1}{2}q\,\vec{B}\times(\vec{\omega}\times\vec{r})$$

- We are assuming $\vec{B}$ is small, so the last term is $\propto B^2$ and negligible

+ In the rotating frame, the charged particle moves in an elliptical orbit (if it is bound to the potential)

+ That means the inertial frame motion is an ellipse that slowly rotates around $\vec{B}$ (slowly b/c B is small)





View from top (excuse poor artwork)

+ The slow rotation of the orbit is precession, which we will see for rigid bodies. When the cause is a magnetic field, it is the Larmor effect. ω = Larmor frequency = $\frac{1}{2}$ cyclotron freq.

— Torque-based analysis? to relate to next unit

• The central $1/r^2$ force has no torque around the origin, but the Lorentz force has

$$d\vec{J}/dt = \vec{\tau} = \vec{r} \times \vec{F} = q\, \vec{r} \times (\vec{v} \times \vec{B}) = q\left[(\vec{r}\cdot\vec{B})\vec{v} - (\vec{r}\cdot\vec{v})\vec{B}\right]$$

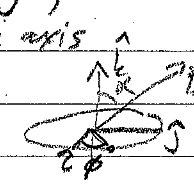
+ For small $\vec{B}$, we can find the torque on a fixed orbit. The small torque then slowly shifts the orbit

+ Torque varies around the fixed orbit. Only the time-average torque ("secular term") leads to gradual orbit changes

• Sample circular orbit

+ For simplicity, choose a circular orbit in x y plane with $\vec{B}$ at angle α from z axis and $\perp$ x axis

+ We can describe particle position by azimuthal angle ϕ



$$\vec{r} = r_0 \cos\phi\, \hat{\imath} + r_0 \sin\phi\, \hat{\jmath}$$

$$\vec{v} = -v_0 \sin\phi\, \hat{\imath} + v_0 \cos\phi\, \hat{\jmath}$$

$$\vec{B} = B_0 \sin\alpha\, \hat{\jmath} + B \cos\alpha\, \hat{k}$$

+ Since $\vec{r}\cdot\vec{v}=0$ for circular motion $\vec{\tau} = q(\vec{r}\cdot\vec{B})\vec{v}$

or $\vec{\tau} = qBr_0v_0 \sin\alpha \sin\phi\,(-\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath})$

On average

$$\langle\vec{\tau}\rangle = -\tfrac{1}{2} qBr_0v_0 \sin\alpha\,\hat{\imath}$$

+ Reminder on the average around a circle

$$\langle \sin^2\phi\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, \sin^2\phi = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, \tfrac{1}{2}(1-\cos 2\phi) = \tfrac{1}{2}$$

$$\langle\cos^2\phi\rangle = \tfrac{1}{2}, \quad \langle \sin\phi\cos\phi\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, \tfrac{1}{2}\sin(2\phi) = 0$$

+ So the orbit starts to tilt, rotating around y axis

• We'd like to write the torque in more physical variables.

+ Note that $\vec{B}\times\vec{J} = m\vec{B}\times(\vec{r}\times\vec{v}) = m\left[(\vec{r}\cdot\vec{B})\vec{v} - (\vec{v}\cdot\vec{B})\vec{r}\right]$

+ The 1st term is $m(\vec{r}\cdot\vec{B})\vec{v} = (m/q)\vec{\tau}$

+ The 2nd term is $-m(\vec{v}\cdot\vec{B})\vec{r} = -mBv_0r_0\sin\alpha\cos\phi(\cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath})$

The time average is

$$-\tfrac{1}{2} mBv_0r_0 \sin\alpha\,\hat{\imath} = (m/q)\langle\vec{\tau}\rangle$$

+ That means $\langle\vec{B}\times\vec{J}\rangle = (2m/q)\langle\tau\rangle$

That is

$$\langle d\vec{J}/dt\rangle = \langle\vec{\tau}\rangle = (q/2m)\langle\vec{B}\times\vec{J}\rangle$$

• Physical Consequences

+ This looks like Lorentz force equation:

$\vec{J}$ moves in circle $\perp \vec{B}$ w/ frequency $qB/2m$ as above

+ Can see $\vec{J}^2$ constant, etc.