Noninertial Reference Frames:

⊙ General Principles

- Preview: Lorentz force / Particle in Magnetic Field
  - The force on a charged particle is
  $$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \quad \text{for charge } q, \vec{E} \text{, } \vec{B} \text{ fields}$$
  - Let's consider $\vec{E} = 0$, $\vec{B} = B\hat{z}$

    + Because $\vec{F} \cdot \vec{v} = 0$, the Lorentz force does no work (conservative in an odd way)
    + Equations of motion are
    $$m\ddot{x} = qB\dot{y}, \quad m\ddot{y} = -qB\dot{x}, \quad m\ddot{z} = 0$$
    + For reasons we'll see below, define $\omega_B = qB/m$

  - So ODEs by coupled ODEs
    + First, $\dot{z} = \text{const} = v_z$
    + Remaining
    $$\ddot{x} = \omega_B \dot{y}, \quad \ddot{y} = -\omega_B \dot{x}.$$

    + If we integrate, we see $\dot{y} = -\omega_B x + \omega_B x_0$
      for some integration constant x_0
    + Then
    $$\ddot{x} + \omega_B^2 x = \omega_B^2 x_0 \Rightarrow x = x_0 + r\cos(\omega_B t + \theta_0)$$
    with r + θ_0 constants
    + Plug back
    $$\dot{y} = -r\omega_B \cos(\omega_B t + \theta_0) \Rightarrow y = y_0 - r\sin(\omega_B t + \theta_0)$$
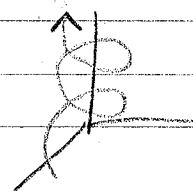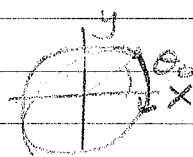 $$\text{constant } y_0$$

    + Path is a helix:
      Circular motion in xy plane (clockwise)
      Constant velocity in $\hat{z}$
      Speed in xy plane constant $= v_\perp$
      and radius is $r = v_\perp / \omega_B$ ⊥ to $\vec{B}$

    $$\omega_B = \text{cyclotron frequency}$$

- Solution by complex numbers
  + Start again with
  $$\ddot{x} = \omega_B \dot{y}, \quad \ddot{y} = -\omega_B \dot{x}$$
  + Define $u = x + iy \Rightarrow \ddot{u} = -i\omega_B \dot{u}$
  + This has solution
  $$\dot{u} = -iv_\perp e^{-i(\omega_B t + \theta_0)} \qquad u = u_0 + (v_\perp/\omega_B) e^{-i(\omega_B t + \theta_0)}$$
  The $-i$ is chosen so position amplitude is real.
  + This breaks down to
  $$x = x_0 + (v_\perp/\omega_B)\cos(\omega_B t + \theta_0), \quad y = y_0 - (v_\perp/\omega_B)\sin(\omega_B t + \theta_0)$$
  as before.

- What if we used a rotating (noninertial) set of axes?

  + We can take them rotating at ω_B clockwise
  $$\vec{\omega} = -\omega_B \hat{k}$$
  + In this frame, particle is still in xy plane, constant velocity in $\hat{z}$
  + Compared to Newton's law, looks like no force. How does this relate to EOM
  $$d\vec{v}/dt = \vec{\omega} \times \vec{v} \quad ?$$

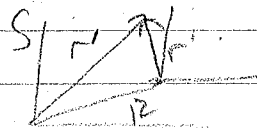- Acceleration of origin (with fixed axis orientation)

  • Suppose frame S is inertial and the origin of S' has position $\vec{R}(t)$ measured w.r.t. S
  + Axes of S' always $\parallel$ to axes of S (assumed)
  + The position of an object measured wrt S' is given by
  $$\vec{r}' = \vec{r}' - \vec{R}$$
  where $\vec{r}$ is measured wrt S
  + So
  $$\vec{v}' = d\vec{r}'/dt = \vec{v}' - d\vec{R}/dt$$
  ie, there is a shift of velocity
  This is true even if S' is inertial (unaccelerating)

+ But if S' accelerates,
$$\vec{a}' = \vec{a} - d^2\vec{R}/dt^2$$

o What about the force measured in the accelerated frame?

+ "Newton's 2nd law" says $\vec{F}' = \vec{F}_{phys} - m\,d^2\vec{R}/dt^2$
the 2nd term on the RHS is a "fictitious force"

+ Example: Consider a ball or stationary pendulum
in accelerating car, bus, elevator, etc.
A ball will seem to accelerate "out of nowhere"
due to the fictitious force as seen by accelerated observer

+ If a frame is uniformly + constantly accelerated,
a freely moving object appears to be
accelerated as if by uniform gravity.
The idea that the gravitational force is fictitious
is the equivalence principle + a key idea of
general relativity.

- Rotating reference frame

 • Setup:
   + There is some set of
   fixed (inertial) axes
   given by unit
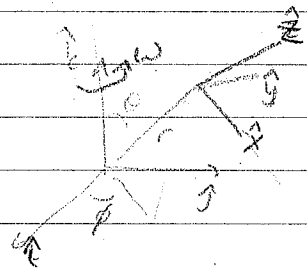
   + We want to work with x, y, z axes with origin
   at spherical coordinates (r, θ, ϕ). These axes
   are rotating around the $\theta = 0$ $\hat{k}$ axis at
   angular frequency ω, ie, $\vec{\omega} = \omega \hat{k}$.

   + To be clear about which axes are which, call
   the rotating unit vectors on the x, y, z axes
   $\hat{x}, \hat{y},$ and $\hat{z}$.

- We want to understand how $\hat{x}, \hat{y}, \hat{z}$ change in time.

  + Consider the case where $\hat{x} = \hat{\theta}$, $\hat{y} = \hat{\phi}$, $\hat{z} = \hat{r}$ at the position of (r, θ, ϕ)

  + The rotation around $\hat{r}$ means $\dot{\phi} = \omega$

  + From a previous homework, we know

$$\hat{x} = \hat{\theta} = \cos\theta(\cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}) - \sin\theta\,\hat{k}$$
$$\hat{y} = \hat{\phi} = -\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}$$
$$\hat{z} = \hat{r} = \sin\theta(\cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}) + \cos\theta\,\hat{k}$$

  + With $\dot{\theta} = 0$, $\dot{\phi} = \omega$, we also know from HW

$$\frac{d\hat{x}}{dt} = \frac{d\hat{\theta}}{dt} = \omega\cos\theta\,\hat{y} = \vec{\omega} \times \hat{x}$$

$$\frac{d\hat{y}}{dt} = \frac{d\hat{\phi}}{dt} = -\omega(\cos\theta\,\hat{x} + \sin\theta\,\hat{z}) = \vec{\omega} \times \hat{y}$$

$$\frac{d\hat{z}}{dt} = \frac{d\hat{r}}{dt} = \omega\sin\theta\,\hat{y} = \vec{\omega} \times \hat{z}$$

- We can use this to determine derivatives of any vector for a rotation

  + Note that we can write any vector $\vec{a}$ as
$$\vec{a} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$$
  at any time

  + The time derivative is $\qquad$ $\overset{\text{previously, we assumed}}{\underset{\text{unit vectors constant}}{\nwarrow}}$

$$\frac{d\vec{a}}{dt} = \left(\dot{a}_x\hat{x} + a_x\left(\frac{d\hat{x}}{dt}\right)\right) + \cdots = \overset{\circ}{\vec{a}} + \vec{\omega} \times \vec{a}$$

  + We use KB notation to mean $\overset{\circ}{\vec{a}}$ is given by derivatives of the components w.r.t. the rotating axes. Meanwhile $d\vec{a}/dt$ is the full derivative including the rotation of $\hat{x}, \hat{y}, \hat{z}$ axes.

+ This also means a different set of axes $\hat{x}'$, $\hat{y}'$, $\hat{z}'$ rotating in the same way has

$$\frac{d\hat{x}'}{dt} = \vec{\omega} \times \hat{x}', \quad \frac{d\hat{y}'}{dt} = \vec{\omega} \times \hat{y}', \quad \frac{d\hat{z}'}{dt} = \vec{\omega} \times \hat{z}'$$

b/c the components wrt $\hat{x}, \hat{y}, \hat{z}$ are constant.

+ Note that this just accounts for rotation + Not motion of the (x, y, z) origin.

+ Because $\vec{\omega} \times \vec{\omega} = 0$, $d\vec{\omega}/dt = \dot{\vec{\omega}}$. The change of the angular velocity is the same in the rotating frame as in an inertial frame.

— Acceleration/Fictitious Forces in Noninertial Frames.

• Relation of velocity between inertial + noninertial frame

+ Our rotating axes above also have a moving origin located at $\vec{R}(t)$ wrt. the inertial frame origin.

+ The inertial frame velocity is

$$\vec{v}' = \frac{d\vec{r}'}{dt} = \frac{d\vec{r}}{dt} + \frac{d\vec{R}}{dt} = \frac{d\vec{R}}{dt} + \dot{\vec{r}} + \vec{\omega} \times \vec{r}'$$

+ The inertial frame acceleration is

$$\vec{a}' = \frac{d^2\vec{r}'}{dt^2} = \frac{d^2\vec{R}}{dt^2} + \ddot{\vec{r}} + \dot{\vec{\omega}} \times \dot{\vec{r}} + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \dot{\vec{r}} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$= \frac{d^2\vec{R}}{dt^2} + \ddot{\vec{r}} + \dot{\vec{\omega}} \times \vec{r} + 2\vec{\omega} \times \dot{\vec{r}} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

+ If the motion of $\vec{R}$ is due to the same rotation around the origin,

$$\frac{d\vec{R}}{dt} = \vec{\omega} \times \vec{R}, \quad \frac{d^2\vec{R}}{dt^2} = \dot{\vec{\omega}} \times \vec{R} + \vec{\omega} \times (\vec{\omega} \times \vec{R})$$

- The rotating observer sees a variety of fictitious forces

  + $\vec{F} \equiv m\ddot{\vec{r}} = \vec{F}'_{(phys)} - m\dot{\vec{\omega}} \times (\vec{R} + \vec{r}) - 2m\vec{\omega} \times \dot{\vec{r}}$
  $$- m\vec{\omega} \times (\vec{\omega} \times (\vec{R} + \vec{r}))$$

  + $\vec{F}'_{(phys)}$ is the actual force
  + $-m\dot{\vec{\omega}} \times (\vec{R} + \vec{r})$ is called the transverse force

  + $-2m\vec{\omega} \times \dot{\vec{r}}$ is the Coriolis force

  + $-m\vec{\omega} \times (\vec{\omega} \times (\vec{R} + \vec{r})) = -m(\vec{\omega} \cdot (\vec{R} + \vec{r}))\vec{\omega} + m\omega^2(\vec{R} + \vec{r})$
  is the centrifugal force

- Back to our magnetic field example, we can choose
  $|\vec{R}| = v/\omega_B$, $\vec{r} = 0$.

  + Inertial frame velocity is $\vec{v}' = \frac{d\vec{R}}{dt} = \vec{\omega} \times \vec{R}$
  So Lorentz force with $\vec{\omega} = -\omega_B \hat{k} = -q\vec{B}/m$
  is
  $$\vec{F}' = q\vec{v} \times \vec{B} = +m\vec{\omega} \times \vec{v}' = +m\vec{\omega} \times (\vec{\omega} \times \vec{R})$$

  + The only fictitious force is centrifugal $-m\vec{\omega} \times (\vec{\omega} \times \vec{R})$
  They exactly cancel!

• Example: Puck on merry-go-round. The puck can move in the $z=0$ plane on a disk rotating with $\vec{\omega} = \omega\hat{z}$ (constant). Let's compare inertial and rotating points of view

+ From an inertial frame, the puck moves in a straight line (once it starts moving). We may allow friction.
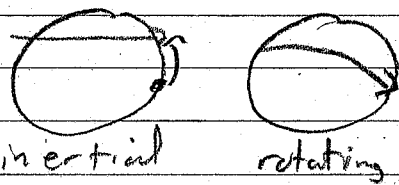
+ In the merry-go-round rotating frame, the apparent force includes centrifugal + Coriolis (origin at merry-go-round axis stationary)

$$\vec{F} = \vec{F}_{friction} + m\omega^2\vec{r} - 2m\vec{\omega}\times\dot{\vec{r}}$$
↳ simplified

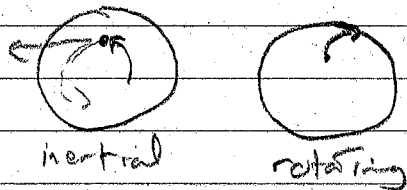Centrifugal = pushes out, Coriolis = push of velocity to right
+

* An inertial observer throws the puck across the turntable

inertial   rotating

In inertial frame, exit point "meets" puck. Rotating observer sees puck bend right

+ A rotating observer lets go of the puck

inertial   rotating

In the rotating frame, the puck moves outwards + right.

+ The puck is sitting at a fixed location on the merry-go-round? What's the minimum coefficient of static friction?

Circular motion

In inertial frame → $m\omega^2\rho\,\hat{\rho} = m\vec{a}_{centrip} = \vec{F}_{friction}, |\vec{F}| \leq \mu_s mg$

$\Rightarrow \mu_s \geq \omega^2\rho/g$

ie, static friction provides centripetal acceleration.

At rest in rotating frame $\vec{F}_{friction} + m\omega^2\rho\,\hat{\rho} = 0$
Same result from balancing friction vs centrifugal.