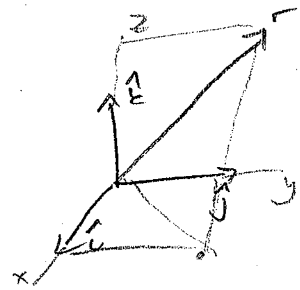
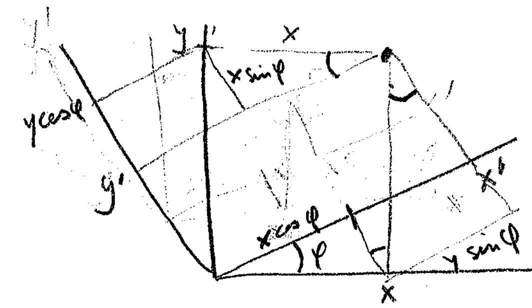
3D Motion

Vector Calculus Review



— Definition of vectors

- As noted, a position vector $\vec{r} = (x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}$
  - + But not all vectors are positions. Velocity, momentum, acceleration are other examples
  - + Vectors add + subtract component by component
    $\vec{a} + \vec{b} = (a_x + b_x)\hat{i} + (a_y + b_y)\hat{j} + (a_z + b_z)\hat{k}$, etc
  - + Multiply by scalars (numbers) on each component.
- In physics, vectors are quantities with components that rotate like position



  - + Consider a rotation around z-axis
    Then $x' = x\cos\varphi + y\sin\varphi$
    $y' = -x\sin\varphi + y\cos\varphi$
  - + In unit vectors $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = x'\hat{i}' + y'\hat{j}' + z'\hat{k}'$
    Can you work out $\hat{i}', \hat{j}', \hat{k}'$ in terms of $\hat{i}, \hat{j}, \hat{k}$?
  - + We can define a matrix R for each rotation such that the components are given by multiplication $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R \begin{bmatrix} x \\ y \\ z \end{bmatrix}$
  - + Any vector's components transform in this manner for a given rotation. That is, $V_{x'} = V_x\cos\varphi + V_y\sin\varphi$, etc.

— Vector Products

- The scalar or dot product: commutes and distributes, and the square of a vector gives the square of its length
  - + Then $|\vec{r}_1 + \vec{r}_2|^2 \equiv r_1^2 + r_2^2 + 2\vec{r}_1\cdot\vec{r}_2 = r_1^2 + r_2^2 + 2r_1r_2\cos\alpha$
    where α = angle between the 2 vectors. by law of cosines
    $\Rightarrow \vec{r}_1 \cdot \vec{r}_2 = r_1 r_2 \cos\alpha$

+ By Pythagorean theorem $\vec{r}^2 = x^2 + y^2 + z^2 \Rightarrow \vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z$

+ These all follow from distributive property for
$$\hat{\imath}^2 = \hat{\jmath}^2 = \hat{k}^2 = 1, \quad \hat{\imath}\cdot\hat{\jmath} = \hat{\imath}\cdot\hat{k} = \hat{\jmath}\cdot\hat{k} = 0.$$

- Cross Product (vector Product)
  + $\vec{a}\times\vec{b}$ = a vector directed according to the right-hand rule: Sweep fingers from direction of $\vec{a}$ to $\vec{b}$ w/right hand, then thumb is along direction of $\vec{a}\times\vec{b}$
  + K&B denotes this product with a wedge $\vec{a}\wedge\vec{b}$, Be careful!
  + The right-hand rule means $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a} \Rightarrow \vec{a}\times\vec{a} = 0$
    It is, however, distributive.
  + We define the action on orthogonal unit vectors
    $$\hat{\imath}\times\hat{\imath} = \hat{\jmath}\times\hat{\jmath} = \hat{k}\times\hat{k} = 0, \quad \hat{\imath}\times\hat{\jmath} = \hat{k}, \quad \hat{\jmath}\times\hat{k} = \hat{\imath}, \quad \hat{k}\times\hat{\imath} = \hat{\jmath}$$
    This means
    $$\vec{a}\times\vec{b} = (a_y b_z - a_z b_y)\hat{\imath} + (a_z b_x - a_x b_z)\hat{\jmath} + (a_x b_y - a_y b_x)\hat{k}$$
    $$= \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z\end{bmatrix}$$
  + If we choose $\vec{a}$ along $\hat{\imath}$ and $\vec{b}$ in the xy plane, we can see
    $|\vec{a}\times\vec{b}| = ab\sin\theta$, where θ = angle between vectors
  + The triple product is cyclicly symmetric $(\vec{a}\times\vec{b})\cdot\vec{c} = (\vec{b}\times\vec{c})\cdot\vec{a}$ etc
  + Vector triple product follows "BAC-CAB rule"
    $$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$$ Note: <u>not</u> associative

— Differentiation
- Suppose we have a vector that depends on time, like an object's position $\vec{r}(t)$
  + Define components w.r.t. fixed orthogonal uniform axes $\hat{\imath}, \hat{\jmath}, \hat{k}$; $\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$
  + Then differentiation applies to components
    $$\frac{d\vec{a}}{dt} \equiv \dot{\vec{a}} = \dot{a}_x(t)\hat{\imath} + \dot{a}_y\hat{\jmath} + \dot{a}_z\hat{k}$$

+ We will discuss non-uniform or moving axes later in term

+ Vector products obey product rule, keeping to order

$$\frac{d}{dt}(\vec{a}\cdot\vec{b}) = \dot{\vec{a}}\cdot\vec{b} + \vec{a}\cdot\dot{\vec{b}}, \quad \frac{d}{dt}(\vec{a}\times\vec{b}) = \dot{\vec{a}}\times\vec{b} + \vec{a}\times\dot{\vec{b}}$$

+ Time integrals are anti-derivatives $\vec{a}(t) = \int dt\, \dot{\vec{a}}(t)$

- Gradient: A function of 3D position $f(\vec{r})$
  + Has partial derivatives $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$
  + Partial derivatives assemble into the gradient

$$\vec{\nabla} f = (\partial f/\partial x)\,\hat{\imath} + (\partial f/\partial y)\,\hat{\jmath} + (\partial f/\partial z)\,\hat{k} \Rightarrow \text{vector function of } \vec{r}.$$

  + The Taylor expansion is

$$f(\vec{r}+\Delta\vec{r}) = f(\vec{r}) + \left(\frac{\partial f}{\partial x}\Delta x + \ldots\right) + \ldots = f(\vec{r}) + \Delta\vec{r}\cdot\vec{\nabla} f + \ldots$$

  so $\vec{\nabla} f$ points in direction of greatest increase of $f(\vec{r})$
  + Chain rule for particle motion $\frac{d}{dt} f(\vec{r}(t)) = \vec{v}(t)\cdot\vec{\nabla} f$.

- Divergence: For a vector function of position $\vec{A}(\vec{r})$
  + Think of $\vec{\nabla}$ as a vector, so divergence is "dot product"

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

  + This is the "net flow of $\vec{A}$ out of a box at $\vec{r}$"
  + The Laplacian of a function is

$$\nabla^2 f \equiv \vec{\nabla}\cdot\vec{\nabla} f = \partial^2 f/\partial x^2 + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

- Curl: vector-valued derivative of a vector
  + Think of it as a cross product

$$\vec{\nabla}\times\vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}$$

  + Curl represents the "rotation" or "vorticity" of $\vec{A}$
  + By commutativity of partial derivatives

$$\vec{\nabla}\times(\vec{\nabla} f) = 0 \quad \text{for any well-defined } f$$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0 \quad \text{"} \quad \vec{A}.$$

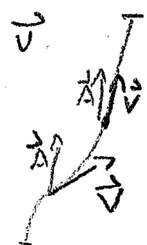
- Product rules work as expected; see KB for a list

- Integrals + Integral theorems
  - Integrals of scalars in 2D + 3D are just repeated integrals (functions)
    + ex $\int_V d^3\vec{x}\, f(\vec{x}) = \int dx \int dy \int dz\, f(x,y,z)$ in Cartesian coords with limits chosen to describe Volume V. Do z integral first in this example, but you can change the order

      This is something like integral of density to get mass
    + You can integrate a vector function one component at a time if the components are for fixed uniform unit vectors (ie, Cartesian)

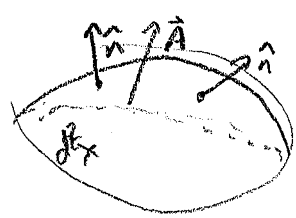
      This is like finding center of mass position
  - Line Integrals of vectors
    + Integral of the tangential component of a vector along a curve
    + Written $\int d\vec{x} \cdot \vec{A}$ implies $= \int dx A_x + \int dy\, A_y + \int dz\, A_z$

      But usually more convenient to parameterize the path, so $d\vec{r} = d\lambda \left(\frac{d\vec{r}}{d\lambda}\right) = (d\,\text{param}) \times (\text{tangent vector})$, As in $d\vec{r} = dt\, \vec{v}$
    + Integral around a closed path denoted $\oint$. Taken in counterclockwise direction.



  - Surface integral of vector
    + We integrate over an area the component of a vector $\perp$ to surface
    + Written as

      $$\int d\vec{S} \cdot \vec{A} = \int d^2x\, (\hat{n} \cdot \vec{A})$$ where $\hat{n}$ is the unit vector $\perp$ the surface at each point



    + Again, integral over a closed surface denoted $\oint$

- Theorems

+ Fundamental theorem of Calculus $\int d\vec{r}\cdot\vec{\nabla}f = \Delta f$

+ Stoke's theorem

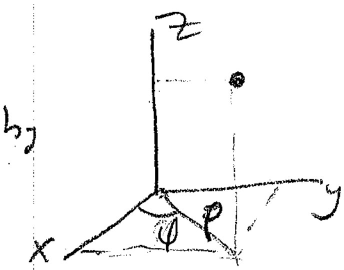
$$\oint d\vec{r}\cdot\vec{A} = \int d\vec{S}\cdot(\vec{\nabla}\times\vec{A})$$ where the surface is <u>any</u> one bounded by the curve w/ $\hat{n}$ chosen by right-hand rule

+ Gauss's theorem

$$\oint d\vec{S}\cdot\vec{A} = \int d^3\vec{x}\ \vec{\nabla}\cdot\vec{A}$$ where $\hat{n}$ points outward and the volume is that enclosed

- Curvilinear Coordinates

- Cylindrical coordinates

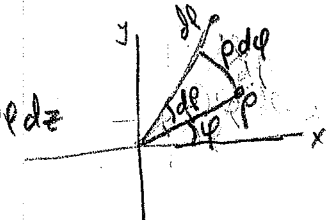


+ These are related to Cartesian coordinates by

$x=\rho\cos\varphi,\quad y=\rho\sin\varphi,\quad z=z$

$\rho^2=x^2+y^2,\quad \tan\varphi=y/x.$

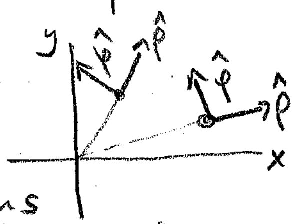
+ The infinitesimal Pythagorean theorem



is $ds^2 = d\rho^2+\rho^2 d\varphi^2+dz^2 \Rightarrow d^3\vec{r} = \rho\, d\rho\, d\varphi\, dz$

using arclength distance

+ There are associated unit vectors (orthogonal)



$\hat{\rho},\ \hat{\varphi},\ \hat{z}$

Note that $\hat{\rho}$ + $\hat{\varphi}$ point in different directions at different locations — they are <u>not</u> uniform

+ The velocity in cylindrical coordinates is

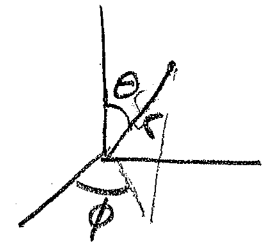
$$\vec{v} = v_\rho\hat{\rho}+v_\varphi\hat{\varphi}+v_z\hat{z} \text{ with } v_\rho=\dot{\rho},\ v_\varphi=\rho\dot{\varphi},\ v_z=\dot{z}$$

+ Based on Pythagorean theorem

$$\vec{\nabla}f = \frac{\partial f}{\partial\rho}\hat{\rho}+\frac{1}{\rho}\frac{\partial f}{\partial\varphi}\hat{\varphi}+\frac{\partial f}{\partial z}\hat{z}$$

Other derivatives are listed in appendix A of KB

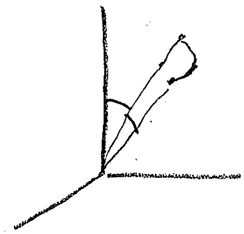
- Spherical polar coordinates



+ Related to Cartesian by

$$x = r\sin\theta\cos\phi,\quad y = r\sin\theta\sin\phi,\quad z = r\cos\theta$$

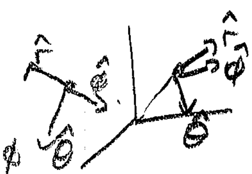
$$r^2 = x^2+y^2+z^2,\quad \tan\phi = y/x$$



+ The infinitesimal distance is

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$$

with volume $\quad d^3\vec{r} = r^2\sin\theta\, dr\, d\theta\, d\phi$



+ The 3 orthogonal unit vectors are

$\hat{r}$ = away from origin

$\hat{\phi}$ = ∥ x-y plane in direction of increasing ϕ

$\hat{\theta}$ = downward along "longitude line". These are <u>not</u> uniform

+ So a velocity is

$$\vec{v} = v_r\hat{r} + v_\theta\hat{\theta} + v_\phi\hat{\phi} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\,\dot{\phi}\hat{\phi}$$

+ Gradient is

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{\phi} \quad \text{etc.}$$